KAPITEL 4 / *CHAPTER 4*[4]
GENERAL SOLUTIONS OF SYSTEMS OF INHOMOGENEOUS EQUATIONS OF HIGH ORDERS OF THE VARIANT OF MATHEMATICAL THEORY OF NON-THIN PLATES

ЗАГАЛЬНІ РОЗВ'ЯЗКИ СИСТЕМ НЕОДНОРІДНИХ РІВНЯНЬ ВИСОКИХ ПОРЯДКІВ ВАРІАНТА МАТЕМАТИЧНОЇ ТЕОРІЇ НЕТОНКИХ ПЛАСТИН

DOI: 10.30890/2709-2313.2022-09-01-014

Introduction

For the calculation of plates and shells are used mainly theories that are based on assumptions and models [1, 2, 4, 5, 9, 11, 12, 21, 23, 25, 26]. Such theories can well describe certain classes of boundary value problems. But these classes need to be outlined for each theory by comparing their solution with the exact one derived from the three-dimensional equations of the theory of elasticity. The exact solution is difficult to come by. Only in some isolated cases it is possible to accurately solve the boundary value problem of the three-dimensional theory of elasticity [6, 14, 18]. To determine the limits of suitability of each theory, you can also compare the results with the solution of the corresponding boundary value problem, which is obtained on the basis of different variants of mathematical theory (MT). Variants of mathematical theories are based on the method of decomposition of the components of the stress-strain state (SSS) into infinite mathematical series [3, 7, 8, 10, 13, 16, 17, 19, 27, 29–35]. The construction of the three-dimensional problem of the theory of elasticity was performed by variational or other methods. The effectiveness of variants of the MT of plates and shallow shells depends on the methodology of construction of basic relations, differential equations (DEs), boundary conditions on the lateral surface, on the exact or approximate satisfaction of boundary conditions on the front faces. To evaluate the effectiveness of variants of MT, comparisons are needed with the exact solution of the three-dimensional theory of elasticity or with reliable variants of MT, which accurately describe the internal SSS and marginal effects.

In [3, 7, 8, 16, 17, 19, 20, 27, 29-35] Legendre polynomials were used to construct the basic equations. Reisner's variational principle [22] was used in [16, 19, 20, 21, 29-35]. In [20], the construction of the basic equations for a transversely isotropic plate, which is obliquely transversely loaded with respect to the median plane, was initiated. The components of displacements were taken in the form of two

[4] ***Authors: Zelensky A. G.***

terms with Legendre polynomials. Interrelated equations were obtained.

In [29–33], a new variant of the MT of plates and shallow shells of arbitrary constant thickness under the action of arbitrary transverse loads was developed. The variant is based on the decomposition of all components of the SSS and boundary conditions on the side surface into infinite mathematical series of Legendre polynomials in the transverse coordinate. Three-dimensional DEs of the theory of elasticity, the semi-inverted Saint-Venan method, the Reissner variational principle are used to reduce the three-dimensional problem of the theory of elasticity to two-dimensional. An important advantage over many other versions of the theory is that the boundary conditions are accurately satisfied on the front faces. The constructed basic dependences and equations of this variant of MT give a real opportunity to obtain analytical solutions of boundary value problems that take into account boundary effects. Numerical results obtained on the basis of the developed MT variant for SSS of a wide class of boundary value problems show high accuracy of the constructed theory in comparison with the exact solution taking into account the first six terms of partial sums for tangential displacements [29, 30]. An increase in the number of terms in the partial sums of mathematical series leads to an increase in the order of inhomogeneous systems of differential equilibrium equations, to the complication of basic equations, and to the complication of finding general solutions.

To date, there are few developed analytical methods for solving inhomogeneous systems of differential equations of high-order equilibrium, which would simplify the finding of partial and general solutions by methods of mathematical physics.

4.1. Statement of the problem and the idea of its solution

A transvepsally isotropic plate of arbitrary constant thickness, which is subjected to arbitrary transverse loading, is considered. The isotropy plane coincides with the median plane. Tangential rectangular coordinates x, y are located in the middle plane, the transverse coordinate z is directed upwards perpendicular to it. Boundary conditions on the side surface can be static, kinematic or mixed.

All SSS components and boundary conditions on the side surface are considered functions of three variables and are represented by infinite mathematical series using Legendre polynomials. If we take into account in tangential displacements components with indices 0, 1, 2,…, N $(u_0, v_0, u_1, .v_1, w_1, ..., u_N, .v_N, w_N)$, where N will be considered an odd natural number, then such an approximation will be called an approximation K0-N (AK0-N). If we take into account components with indices 1, 3 ,…, N $(u_1, .v_1, w_1, ..., u_N, .v_N, w_N)$, then such an approximation will be called the approximation K13… N (AK13 N).

The components of displacements in the K0-N approximation are represented by Legendre polynomials in the form (1):

$$U(x,y,z) = \sum_{k=0}^{N} P_k(2z/h)u_k(x,y), (U,u \to V,v); W(x,y,z) = \sum_{k=1}^{N} P_{k-1}(2z/h)w_k(x,y). \quad (1)$$

The problem in the constructed variants of MT is to solve systems of DEs of equilibrium of high orders and to construct their general solutions. The main mathematical difficulty is finding general solutions of inhomogeneous systems of equations of equilibrium of high orders, including finding their partial solutions. Difficulties increase if the transverse load is intermittent or local.

In many works on the theory of plates and shells, general and partial solutions were determined directly from the initial systems of equilibrium equations. In particular, methods of integral transformations were used to find partial solutions [24, 28]. In some cases, it is almost impossible to use these methods due to the complex and cumbersome inverse integral transformations, as it is necessary to find cumbersome integrals with parameters that are not listed in the known literature. The method of direct solution of the initial systems of equations of equilibrium of high orders has led to significant difficulties [28].

The obtained systems of equilibrium equations of the constructed MT variant have a rather high order. These are systems of equations with respect to the components of displacements. In the approximation K0-3 the order of the system is twenty-second. In the approximation K0-5 the order of the system is thirty-fourth. Therefore, the direct analytical solution of such systems of equations is associated with great difficulties.

A new method of integrating systems of high-order DEs is that the initial system of high-order equilibria was reduced by algebraic, differential, and operator transformations to convenient (solvable) inhomogeneous high-order differential equations. These equations were reduced by the operator method to inhomogeneous second-order DEs. This significantly reduced the difficulty of finding general solutions of the initial system of high-order equilibrium equations. Especially if the transverse load was intermittent, or concentrated or local. General solutions of inhomogeneous second-order DEs could be found by various known methods of mathematical physics. The general solution of the initial system of differential equilibrium equations was found through the general solutions of the second-order equations. SSS was determined from the relevant dependencies based on the solutions of the initial DE systems.

In [15] the inhomogeneous differential equation of the fourth order is reduced to two inhomogeneous Helmholtz equations. Integral transformations were not applied to the obtained equations. In [24], a partial solution of the fourth-order equilibrium differential equations of the theory of thin isotropic spherical shells of small curvature was found by direct application of the Hankel integral transformation. Even for low-order DE, this led to sufficient mathematical complications.

The idea of the developed method is to simplify the definition of partial and general solutions of high-order DEs. Simplification of the search for these solutions is achieved by using methods of reduction of inhomogeneous DEs of high orders to inhomogeneous equations of the second order. The work is a development and generalization of researchs [30, 33-35].

4.2. Basic equations in the approximation K0-N

This section presents the components of SSS, DEs of equilibrium, boundary conditions. The final DEs in the K0-N approximation are derived. General solutions of the system of differential equilibrium equations are obtained.

4.2.1. Components of the stress-strain state.

The components of the displacements are depicted in the form (1).

Stress in the plate:

$$\sigma_{xz}(x,y,z)=\sum_{n=0}^{N+1}P_n(2z/h)t_{xn}(x,y);\sigma_{yz}(x,y,z)=\sum_{n=0}^{N+1}P_n(2z/h)t_{yn}(x,y);$$

$$\sigma_z(x,y,z)=\sum_{n=0}^{N+2}P_n(2z/h)s_{zn}(x,y); \tag{2}$$

$$\sigma_x(x,y,z)=\sum_{n=0}^{N+2}P_n(2z/h)s_{xn}(x,y),\ (x,y);\ \sigma_{xy}(x,y,z)=\sum_{n=0}^{N}P_n(2z/h)t_{yxn}(x,y).$$

In formulas (2), the functions $t_{x\,n},...,t_{y\,x\,n}$ have the following form:

$$t_{x\,n}(x,y)=\sum_{i=1,3}^{N}(h_{0\,n\,i}w_{i,x}+l_{0\,n\,i}u_i),\ (n=0,2,...,N+1);$$

$$t_{x\,n}(x,y)=\sum_{i=2,4}^{N-1}(h_{0\,n\,i}w_{i,x}+l_{0\,n\,i}u_i),\ (n=1,3,...,N);$$

$$s_{z\,n}(x,y)=\sum_{i=2,4}^{N-1}p_{n\,i}w_i+\sum_{i=0,2}^{N-1}g_{n\,i}\varphi_i+g_{n\,p}p\,,\varphi_i=u_{i,x}+v_{i,y},\ (n=0,2,...,N+1);$$

$$s_{z\,n}(x,y)=\sum_{i=3,5}^{N}p_{n\,i}w_i+\sum_{i=1,3}^{N}g_{n\,i}\varphi_i+g_{n\,q}q\,,\ (n=1,3,...,N+2); \tag{3}$$

$$s_{xn}(x,y)=d_0(u_{n,x}+\nu\, v_{n,y})+d_{10}s_{zn}\,,\ (n=0,1,...,N)\,;$$

$$s_{x\,n}(x,y)=d_{10}s_{z\,n}\,,\ (n=N+1,N+2)\,,$$

$$t_{yxn}(x,y)=G(u_{n,y}+v_{n,x}),\ (n=0,1,...,N)\,,$$

where h, l, p, g with indices - mechanical and geometric parameters; d_0, d_{10}, G - mechanical parameters of the plate material. Mechanical and geometric parameters will be called constant.

4.2.2. Differential equations of equilibrium.

Let us represent the equilibrium equation through $u_0,v_0,u_1,v_1,w_1,...,u_N,v_N,w_N$, taking into account the expressions for $t_{x\,i},...,t_{y\,x\,i}$ according to (3), and performing some mathematical transformations.

In the K0-N approximation, the system of differential equilibrium equations is of the order of (6N+4). It is shown that this system of equations is divided into two separate independent systems of equations. One system describes the SSS of a plate with symmetrical deformation relative to the median plane. It occurs under a symmetrical load relative to the median plane, which is applied both on the front faces and on the side surface.

Another system of equations describes SSS for obliquely symmetric deformation. This is a purely bending deformation. It occurs when the obliquely symmetrical load relative to the median plane, which is applied both on the front faces and on the side surface. In all relations and equations it is necessary to take into account only those terms in the components of displacements that are taken into account in the partial sums of mathematical series for tangential components of displacements (1). This also applies to boundary conditions.

It is established that in the approximation K0-N the system of differential equations for obliquely symmetric loading has the order of 3 (N+1), and for the symmetric one - of the order of (3N+1).

DE system describing obliquely symmetric deformation (AK13… N):

$$\sum_{j=1,3}^{N}(L_{iuj}u_j + L_{ivj}v_j + L_{iwj}w_j) = L_{iq}(q(x,y)),\ (i=1,2,...,3(N+1)/2). \quad (4)$$

DE system describing symmetrical deformation (AK02… (N-1)):

$$\sum_{j=0,2}^{N-1}(M_{iuj}u_j + M_{ivj}v_j) + \sum_{j=2,4}^{N-1}M_{iwj}w_j = M_{ip}(p(x,y)),\ (i=1,2,...,(3N+1)/2). \quad (5)$$

In the systems of equations (4), (5) M_{ij}, L_{ij} - differential operators not higher than the 2nd order; $M_{ip}(p)$, $L_{iq}(q)$ - transverse load functions; these operators and functions depend on the mechanical and geometric parameters of the plate.

4.2.3. Boundary conditions.

Boundary conditions follow from the Reisner variation equation and have the form:

$$\int_{(s)}\{\sum_{j=0,1}^{N}\frac{h}{2j+1}((s_{xj}l_x + t_{yxj}l_y - x_{sj})\delta u_j + (t_{yxj}l_x + s_{yj}l_y - y_{sj}))\delta v_j + \\ + \sum_{j=0,1}^{N-1}\frac{h}{2j+1}(t_{xj}l_x + t_{yj}l_y - z_{sj})\delta w_{j+1}\}ds = 0. \quad (6)$$

Different boundary conditions follow from equation (6).

4.2.4. Oblique symmetric deformation of the plate

The methodology of reduction of systems of inhomogeneous equations of equilibrium of high orders to inhomogeneous equations of the second order is given for obliquely symmetric deformation of a plate.

4.2.4.1. Transformation of systems of differential equilibrium equations.

We transform the system of differential equations of equilibrium of skew-symmetric deformation (4). Analyzing the structure of the operators of the system of equations (4), we can reduce this system by mathematical transformations to the following form:

$$\sum_{j=1,3}^{N}(a_{Niuj}u_j + a_{Ni\varphi j}\varphi_{j,x} + a_{Niwj}w_{j,x}) + a_{N1}\psi_{i,y} = \beta_{ui}q_{,x},\ (\delta u_i); \\ \sum_{j=1,3}^{N}(a_{Niuj}v_j + a_{Ni\varphi j}\varphi_{j,y} + a_{Niwj}w_{j,y}) - a_{N1}\psi_{i,x} = \beta_{ui}q_{,y},\ (\delta v_i),\ (i=1,3,...,N); \quad (7)$$

$$\sum_{j=1,3}^{N}(b_{N1\varphi j}\varphi_j + b_{N1wj}\nabla^2 w_j) = \beta_{w1}q, \, (\delta w_1);$$
$$\sum_{j=1,3}^{N} b_{Ni\varphi j}\varphi_j + b_{Niwj}\nabla^2 w_1 + \sum_{j=3,5}^{N}(c_{Niwj} + b_{Niwj}\nabla^2)w_j = \beta_{wi}q, \, (\delta w_i), (i = 3,5,,N), \tag{8}$$

where $\psi_j(x, y)$ - vortex functions:

$$\psi_j(x, y) = u_{j,y} - v_{j,x}, \ \ (j = 1,2,..., N);$$

$\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ – Laplace operator; a, b, c s with indices are constants. Variations in parentheses next to the equations indicate the variation in which the equation is obtained.

The number of DE (7) is equal to (N+1), and DE (8) (N+1) / 2, The number of DE system (7), (8) is equal to 3 (N+1) / 2. The same number of unknown functions $u_1, v_1, w_1, u_3, v_3, w_3, ..., u_N, v_N, w_N$. We number the equation as follows: DE obtained for δu_1 is the first equation, for δv_1 is the second equation, for δu_3 is the third equation, and so on.

To obtain a system of equations describing the vortex boundary effect, the following steps from DE (7) must be performed. The first equation is differentiated by y, the second equation by x and one equation is subtracted from another. The third and fourth equations are similarly transformed, and so on. A homogeneous system of order DE with respect to the vortex functions $\psi_j, (j = 1,3,..., N)$ is obtained:

$$\sum_{j=1,3}^{N} H_{ij}\psi_j(x, y) = 0; \ (i = 1, 3,..., N), \tag{9}$$

where H_{ij} $(i = j)$ are second-order differential operators; and $H_{ij}(i \neq j)$ zero-order operators:

$$H_{ii} = (h_{i2\psi}\nabla^2 + h_{i0\psi}), \ (i = 1,3,..., N); \ H_{ij} = h_{ij0\psi}, \ (i \neq j). \tag{10}$$

In (10) h with indices are constant parameters, $h_{ij0\psi} = h_{ji0\psi}$. The differential matrix of the system of equations (9) is symmetric: $H_{ij} = H_{ji}$ $(j = 1,3,..., N), \ (i \neq j)$.

We now obtain a DE system that does not contain vortex functions and describes internal SSS with a potential marginal effect. Let's convert equation (7) again. Differentiate the first equation for the variable x, the second equation for the variable y and add them; we differentiate the third equation by x, the fourth equation by y and add them, and so on. We obtain the following DE system:

$$\sum_{j=1,3,}^{N}((d_{i\varphi j0}+d_{i\varphi j2}\nabla^2)\varphi_j+d_{iwj2}\nabla^2 w_j)=\beta_{ui}\nabla^2 q,\ (i=1,3,...,N), \tag{11}$$

where d with indices - constant values, which are determined by the parameters DE (7).

From (8) and (11) we obtain an inhomogeneous DE system of order 2 (N+1) with respect to the $w_j(x,y)$ functions:

$$\sum_{j=1,3}^{N}\Pi_{ij}w_j(x,y)=\Pi_{iq}q(x,y)\ \ (i=1,3,...,N), \tag{12}$$

where Π_{ij} are fourth-order differential operators. Π_{iq} – second-order differential operators. All operators in (12) depend on mechanical and geometric parameters. They are found to look like this:

$$\Pi_{11}=\mu_{114}\nabla^4;\ \Pi_{1j}=\mu_{1j4}\nabla^4+\mu_{1j2}\nabla^2,\ \Pi_{j1}=\mu_{j14}\nabla^4+\mu_{j12}\nabla^2,\ (j=2,3...,N);$$

$$\Pi_{ij}=\mu_{ij4}\nabla^4+\mu_{ij2}\nabla^2+\mu_{ij0},(i,j=3,5,...,N;i=j,i\neq j);\Pi_{iq}=\mu_{i2}\nabla^2+\mu_{i0}; \tag{13}$$

$\mu_{114},...\mu_{i0}$ – сталі.

Differential equations (9) and (12) will be called solvable DE of skew-symmetric deformation. From these equations are the functions $\psi_j(x,y)$, $w_j(x,y)$, $(j=1,3,...,N)$. The $\varphi_j(x,y)$ functions are determined from equations (8). Functions $u_j(x,y)$, $v_j(x,y)$, are from equations (7) and are expressed through $\varphi_j(x,y)$, $w_j(x,y)$, $\psi_j(x,y)$. The stresses are according to formulas (2) and (3).

4.2.4.2. General solutions of differential equations of vortex boundary effect.

We reduce the system of equations of the vortex boundary effect (9) to the convenient DE. We transform the system (9) by the operator method. Let us represent the required functions $\psi_j(x,y)$ through the new function $\psi(x,y)$ as follows:

$$\psi_j(x,y)=H_{1j}^0\psi(x,y),\ (j=1,3,...,N;\ N\geq 3), \tag{14}$$

where H_{1j}^0 are the adjuncts of the differential determinant H_0 of the system (9).

The system of equations (9), taking into account (14), will be reduced to the definition of the function $\psi(x,y)$ from a homogeneous DE order (N+1).

$$H_0\psi(x,y)\equiv(\nabla^2-r_1)(\nabla^2-r_2)\cdot...\cdot(\nabla^2-r_{(N+1)/2})\psi(x,y)=0, \tag{15}$$

where r with indices – parameters. Equation (15) will be called the defining differential equation of the vortex boundary effect.

The general solution DE (15) is represented as:

$$\psi(x,y)=\sum_{i=1}^{(N+1)/2}\psi^{(i)}(x,y)\,, \tag{16}$$

where $\psi^{(i)}(x,y)$ are the general solutions of the Helmholtz differential equations

$$(\nabla^2-r_i)\,\psi^{(i)}(x,y)=0,\ \ (i=1,2,...,(N+1)/2;\ N\geq 1)\,. \tag{17}$$

General solutions for vortex functions $\psi_j(x,y)$ are obtained on the basis of (14) –(17):

$$\psi_j(x,y)=H_{1j}^0\sum_{i=1}^{(N+1)/2}\psi^{(i)}(x,y)\,. \tag{18}$$

4.2.4.3. General solutions of differential equations of internal stress state and potential boundary effect.

Consider the system of equations (12), which describes the internal SSS and the potential marginal effect. We represent the functions w_j through the new required functions $\Phi_k(x,y)$ by the operator method:

$$w_j(x,y)=\sum_{k=1,3}^{N}\Pi_{k\,j}^0\Phi_k(x,y),(\,j=1,\,3,...,N;\ N\geq 3), \tag{19}$$

where $\Pi_{k\,j}^0$ are adjuncts of the system determinant (12).

On the basis of (12), (13), (19) we obtain a convenient (determining) system of DE with respect to functions $\Phi_k(x,y)$, which after factorization of the left parts will look like:

$$D_0\,D_0\,D_1\,D_2\cdot...\cdot D_{(N-1)}\Phi_k\,(x,y)=a_{k\,0}\,D_{k\,0}\,q(x,y);\quad k=1,3,...,N;\ N\geq 3, \tag{20}$$

where

$D_0=\nabla^2;\ D_i=\nabla^2-s_i;D_{k0}=\nabla^2-s_{k0};\ \ i=1,2,...,N-1;\ \ s_i,s_{k0},a_{k0}-$ parameters.

The system of differential equations (20) will be called hereinafter the determining system of internal SSS and potential boundary effect.

The DE system (20) is more convenient than (12), because the left parts of this system are the same. Forms of general solutions of the DE system (20) are obtained in the form of:

$$\Phi_k(x,y)=\Phi_{k\,0}(x,y)+\Phi_{k\,r}(x,y),\ \ (k=1,3,...,N), \tag{21}$$

where $\Phi_{k0}(x,y)$ are the general solutions of the corresponding homogeneous DEs of the system (20), $\Phi_{kr}(x,y)$ are the partial solutions of the inhomogeneous DEs (20). Since the homogeneous DE systems (20) are the same, we can put $\Phi_{k0}(x,y)\equiv 0$, ($k=3,5,...,N$) without increasing the order of this DE system. Then the general solutions (20) taking into account (21) will take the form:

$$\Phi_1(x,y)=\Phi_{1B}(x,y)+\Phi_{1\Pi}(x,y)+\Phi_{1r}(x,y);\ \ \Phi_k(x,y)=\Phi_{kr}(x,y),\ (k=3,5,...,N). \quad (22)$$

In formulas (22): Φ_{1B} is the general solution of the biharmonic equation $\nabla^4\Phi_1=0$.

$\Phi_{1\Pi}$ is the general solution of a homogeneous DE of order 2 (N-1):

$$D_1 D_2\cdot...\cdot D_{(N-1)}\Phi_{1\Pi}(x,y)=0; \quad (23)$$

$\Phi_{kr}(x,y)$ are partial solutions of inhomogeneous DEs (20).

The general solution $\Phi_{1\Pi}$ DE (23) is represented as:

$$\Phi_{1\Pi}=\sum_{j=1,2}^{N-1}\Phi_{1\Pi j}(x,y), \quad (24)$$

where $\Phi_{1\Pi j}$ are the general solutions of the equations $(\nabla^2-s_j)\Phi_{1\Pi j}=0,\ (j=1,2,...,N-1)$.

The potential boundary effect is described by a homogeneous DE (23). Internal SSS is determined by the sum of the general solution Φ_{1B} of the biharmonic equation and the partial solutions Φ_{kr} $(k=1,3,...,N)$ of the inhomogeneous DE (20) of order 2 (N+1). Thus, the equations of domestic SSS and potential boundary effect are separated.

The general solutions of the system of equations (12) based on (19), (22), (24) are as follows:

$$w_j(x,y)=\Pi^0_{1j}(\Phi_{1B}(x,y)+\Phi_{1\Pi}(x,y))+\sum_{k=1,3}^{N}\Pi^0_{kj}\Phi_{kr}(x,y),\ (j=1,3,...,N). \quad (25)$$

The components of displacements $u_k,v_k,(k=1,3,...,N)$ are determined from DE (7):

$$u_k(x,y)=\sum_{i=1,3}^{N}(\lambda_{k\varphi i}\varphi_{i,x}+\lambda_{k\psi i}\psi_{i,y}+\lambda_{kwi}w_{i,x})+\lambda_{kq}q_{,x},$$
$$(u_k,\varphi_{,x},\psi_{,y},w_{,x},q_{,x}\to v_k,\varphi_{,y},-\psi_{,x},w_{,y},q_{,y}), \quad (26)$$

where $\varphi_i(x,y)=\lambda_{i1}\nabla^2 w_1+\sum_{k=3,5}^{N}(\lambda_{ik}\nabla^2+\lambda'_{ik})w_k+\lambda_{iq}q,\ (i=1,3,...,N)$, λ with indices -

parameters.

Stresses are determined on the basis of the corresponding formulas (2), (3) through the components of displacements (25), (26).

4.3. Reduction of inhomogeneous differential equations to inhomogeneous second-order equations

Consider DE of order $2n$:

$$(A_n\nabla^{2n} + A_{n-1}\nabla^{2(n-1)} + \ldots + A_0)\Phi(x,y) = f(x,y), \qquad (27)$$

where $A_i \neq 0$ is a constant value, $f(x,y)$ is a known function, $\Phi(x,y)$ is a desired function.

Equation (27) can always be reduced to an inhomogeneous equation of the following form:

$$(\nabla^2 - k_1)(\nabla^2 - k_2)\cdot\ldots\cdot(\nabla^2 - k_i)\cdot\ldots\cdot(\nabla^2 - k_n)\Phi(x,y) = f(x,y), \qquad (28)$$

where k_i $(i = 1,2,\ldots,n)$ - parameters, roots of the corresponding characteristic equation.

The general solutions of homogeneous equations corresponding to equation (28) are expressed through the general solutions of the Helmholtz equations.

The general solutions of inhomogeneous differential equations are represented as the sum of the general solutions of the corresponding homogeneous equations and the partial solutions of the inhomogeneous equations.

In the next section we consider some partial cases of equations (20) and (28).

In what follows, the general solutions of inhomogeneous differential equations of high orders are expressed through the general solutions of inhomogeneous equations of the second order. Therefore, in Section 3.1 we provide information on general solutions of inhomogeneous second-order differential equations.

4.3.1. Inhomogeneous second-order differential equations.

1). Differential equations of the form:

$$D_0\Phi(x,y) = f(x,y), (D_0 = \nabla^2). \qquad (29)$$

The general solution of the Poisson differential equation (29): is

$\Phi(x,y) = f_{00}(x,y) + f_{0r}(x,y)$, where $f_{00}(x,y)$ is the general solution of the corresponding homogeneous DE, determined by the Lagrange method:

$$f_{00}(x,y) = F_1(y - ix) + F_2(y + ix), \ (i = \sqrt{-1}),$$

where F_1, F_2 are arbitrary functions of the corresponding arguments.

The partial solution f_{0r} DE (29) is found by the Lagrange method using the auxiliary equation and has the following form:

$$f_{0r}(x,y) = \frac{1}{D + iD'}\frac{1}{D - iD'}f(x,y) = \frac{1}{D + iD'}q_1(x,y) = q_2(x,y),$$

where

$$D = \partial / \partial x; \ D' = \partial / \partial y; \ \partial^2 / \partial x^2 + \partial^2 / \partial y^2 = (D + iD')(D - iD');$$

$$q_1(x,y) = \int f(x, c - ix)dx\Big|_{c = y + ix}; q_2(x,y) = \int q_1(x, c + ix)\,dx\Big|_{c = y - ix};$$

c under integrals is considered a constant value.

2). Inhomogeneous Helmholtz differential equations:

$$D_i \Phi(x,y) = f(x,y); D_i = \nabla^2 - s_i, \ i = 1,2,..., \tag{30}$$

where s_i are constant values.

The partial solution $f_{ir}(x,y)$ of equation (30) is defined as follows:

$$f_{ir}(x,y) = \frac{f(x,y)}{D_i}.$$

The general solution of a homogeneous DE corresponding to equation (30) in the class of exponential functions with separated variables has the form:

$$f_{i0}(x,y) = \sum_n C_{ni} \exp(a_{ni}x + b_{ni}y), \ (n = 1,2,3,...),$$

where C_{ni} is arbitrary steels; a_{ni} and b_{ni} are constant values that satisfy the $a_{ni}^2 + b_{ni}^2 = s_i$ equation.

4.3.2. Heterogeneous differential equations of the fourth order

1). Differential equations of the form:

$$D_0\, D_0\, \Phi(x,y) = f(x,y) \tag{31}$$

General solution DE (31): $\Phi(x,y) = f_{000}(x,y) + f_{00r}(x,y)$.

The general solution $f_{000}(x,y)$ of homogeneous DE $D_0\, D_0 \Phi(x,y) = 0$ has the form:

$$f_{000}(x,y) = F_1(y-ix) + F_2(y+ix) + xF_3(y-ix) + xF_4(y+ix),$$

where $F_1,...,F_4$ are arbitrary functions from the corresponding variables.

The partial solution DE (31) is as follows:

$$f_{00r}(x,y) = \frac{1}{D+iD'}\frac{1}{D-iD'}f_{0r}(x,y) = \frac{1}{D+iD'}q_3(x,y) = q_4(x,y),$$

where

$$q_3(x,y) = \int f_{0r}(x,c-ix)dx\Big|_{c=y+ix}; q_4(x,y) = \int q_3(x,c+ix)dx\Big|_{c=y-ix}.$$

Partial solutions of the inhomogeneous DE (31) can also be found by sequential integration of inhomogeneous equations of the second order:

$$D_0 f_1(x,y) = f(x,y);\ D_0\Phi(x,y) = f_1(x,y).$$

Here, partial solutions can be used as solutions.

2). Differential equations of the form:

$$D_i D_i \Phi(x,y) = f(x,y),\ (i = 1,2,....). \qquad (32)$$

The partial solution $f_{iir}(x,y)$ of equation (32) is determined by the sequential integration of inhomogeneous Helmholtz differential equations:

$$D_i f_i(x,y) = f(x,y),\ D_i\Phi(x,y) = f_{ir}(x,y). \qquad (33)$$

where $f_{ir}(x,y)$, f_{iir} – are the partial solutions of the first and second equations (33).

The general solution f_{ii0} of the homogeneous equation, which corresponds to equation (32), in the class of exponential functions has the form:

$$f_{ii0}(x,y) = \sum_n C_{ni}\exp(a_{ni}x + b_{ni}y) + x\sum_n D_{ni}\exp(c_{ni}x + d_{ni}y),\ (n = 1,2,3,...),$$

where C_{ni}, D_{ni} is arbitrary steels; $a_{ni}, b_{ni}, c_{ni}, d_{ni}$ - various constants that satisfy the equation: $a_{ni}^2 + b_{ni}^2 = s_i, c_{ni}^2 + d_{ni}^2 = s_i$.

3). Differential equations of the form:

$$D_i D_j \Phi(x,y) = f(x,y),\ (i \neq j;\ i,j = 0,1,2,....). \qquad (34)$$

The partial solution $f_{ijr}(x,y)$ DE (34) is found by the operator method:

$$f_{ijr}(x,y) = \frac{f(x,y)}{D_i D_j} = \frac{1}{D_j - D_i}\left(\frac{f}{D_i} - \frac{f}{D_j}\right) = \frac{1}{s_{ij}}(f_{ir} - f_{jr}),$$

where $f_{ir}(x,y), f_{jr}(x,y)$ are the partial solutions of the corresponding inhomogeneous Helmholtz equations.

Thus, partial solutions of inhomogeneous fourth-order DEs (32) and (34) are expressed in terms of partial solutions of second-order inhomogeneous equations.

The general solution $f_{i\,j\,0}$ of the homogeneous equation corresponding to equation (34) is expressed in terms of the general solutions of the Helmholtz equations. In the class of exponential functions $f_{i\,j\,0}$ has the following form:

$$f_{i\,i\,0}(x,y)=\sum_n C_{ni}\exp(a_{ni}x+b_{ni}y)+\sum_n D_{ni}\exp(c_{ni}x+d_{ni}y),\ (n=1,2,3,...).$$

where C_{ni},D_{ni} is arbitrary steels; $a_{ni},b_{ni},c_{ni},d_{ni}$ - constants that satisfy the equation: $a_{ni}^2+b_{ni}^2=s_i, c_{ni}^2+d_{ni}^2=s_j$.

In the following sections, we present partial solutions of inhomogeneous equations of order above the fourth.

4.3.3. Differential equations of the sixth order

Differential equations of the form:

$$D_i\,D_j^2\,\Phi(x,y)=f(x,y),\ (i\neq j;\ i,j=0,1,2,....). \tag{35}$$

Partial solution of this equation:

$$\Phi_r(x,y)=\frac{f(x,y)}{D_i\,D_j^2}=\frac{1}{s_{ij}^2}\left(\frac{f}{D_i}-\frac{f}{D_j}\right)-\frac{1}{s_{ij}D_j^2}=\frac{1}{s_{ij}^2}(f_{ir}-f_{jr})-\frac{1}{s_{ij}}f_{jjr}. \tag{36}$$

Partial solutions of this equation will be needed in the future to obtain partial solutions of inhomogeneous equations of higher orders.

The partial solution (36) of DE (35) is also expressed in terms of the partial solutions of inhomogeneous equations of the second order, taking into account the partial solution of the inhomogeneous equation (34). The general solution of the homogeneous equation corresponding to DR (35) is expressed in terms of the general solutions of the Helmholtz equations.

4.3.4. Inhomogeneous differential equations of the eighth order.

1). Differential equations of the form:

$$D_1\,D_1\,D_1\,D_1\,\Phi(x,y)=f(x,y). \tag{37}$$

Equations (37) are found in the variant of the mathematical theory of thick plates on an elastic basis.

The partial solution $\Phi_r(x,y)$ of this equation is a successive finding of the partial solutions of the following equations:

$$D_1 f_3(x,y) = f(x,y); D_1 f_2(x,y) = f_{3r}(x,y); D_1 f_1(x,y) = f_{2r}(x,y); D_1 \Phi(x,y) = f_{1r}(x,y).$$

2). Differential equations of the form:

$$D_1\, D_1\, D_2\, D_2\, \Phi(x,y) = f(x,y). \qquad (38)$$

Partial solution of equation (38):

$$\Phi_r(x,y) = \frac{1}{s_{12}^2}(f_{11r} + f_{22r} - \frac{2}{s_{12}}(f_{1r} - f_{2r})).$$

Therefore, the general and partial solutions of the inhomogeneous equations of Section 3.4, taking into account the previous paragraphs, are also expressed through the general and partial solutions of the inhomogeneous equations of the second order.

4.3.5. Inhomogeneous differential equations of the twelfth order

1). Differential equations of the form:

$$D_0\, D_0\, D_1\, D_2 D_3\, D_4 .\Phi_k(x,y) = a_{k0}\, D_{k0}\, f(x,y), (k = 1,3,5)\ , \qquad (39)$$

where D_{k0} $(k = 1,3,5)$ is the Helmholtz differential operator.

The system of differential equations of internal SSS of plates with potential boundary effect at obliquely symmetric deformation in the K135 approximation is reduced to such equations. In [34], partial solutions of Φ_{kr} of equations (39) are obtained. They look like:

$$\Phi_{kr}(x,y) = a_{k0} D_{k0}(\frac{(f_{1r} - f_{0r})}{s_1^2 s_{12} s_{13} s_{14}} + \frac{(f_{2r} - f_{0r})}{(s_2^2 s_{21} s_{23} s_{24})} + $$
$$ + \frac{(f_{3r} - f_{0r})}{(s_3^2 s_{31} s_{32} s_{34})} + \frac{(f_{4r} - f_{0r})}{(s_4^2 s_{41} s_{42} s_{43})} + \frac{f_{00r}}{(s_1 s_2 s_3 s_4)}), \qquad (40)$$

where $f_{0r}(x,y), f_{00r}(x,y), f_{ir}(x,y)$ $(i = 1,2,3,4)$ are the partial solutions of the corresponding inhomogeneous equations.

Thus, the partial solutions (40) of inhomogeneous DE (39) are expressed by the differential operator from the linear combination of partial solutions of inhomogeneous DE of the second order.

The general solutions of DE (39) are determined by the sum of:

$$\Phi_k(x,y) = f_{000}(x,y) + \sum_{i=1}^{4} f_{i0}(x,y) + \Phi_{kr}(x,y),$$

where $f_{000}(x,y)$ is the general solution of the biharmonic DE; $f_{i0}(x,y)$ – general solutions of DE Helmholtz $(\nabla^2 - s_i) f_i(x,y) = 0,\ (s_i - const)$.

2). Differential equations of the form:

$$D_0^2\, D_1^4 \Phi(x,y) = f(x,y)\ . \tag{41}$$

Equations of the form (41) describe the internal SSS with a potential edge effect with four times the root s_1 of the characteristic equation in the K135 approximation.

The partial solution of equation (41) has the form:

$$\Phi_r(x,y) = \frac{1}{s_1^2}\left(-\frac{4}{s_1^3}(f_{1r} - f_{0r}) + \frac{1}{s_1^2}(f_{00r} + f_{11r}) - \frac{2}{s_1} f_{111r} + f_{1111r}\right).$$

Thus, the general solutions of inhomogeneous DEs of the twelfth order, taking into account the general solutions of the corresponding homogeneous equations and partial solutions of inhomogeneous equations, are also expressed through the general solutions of inhomogeneous DEs of the second order.

The system of inhomogeneous differential equilibrium equations, which describes symmetric deformation, is transformed similarly. General solutions are also expressed through general solutions of inhomogeneous second-order equations.

It should be noted that in [35] we obtained partial solutions of inhomogeneous differential equations of the form:

$$D_0\, D_0\, D_1\, D_2 \Phi(x,y) = f(x,y);\ \ D_0\, D_0\, D_1\, D_1 \Phi(x,y) = f(x,y);$$

$$D_0\, D_0\, D_1\, D_2 D_3\, D_4 \Phi(x,y) = f(x,y);\quad D_0^2\, D_1^2\, D_2^2 \Phi(x,y) = f(x,y)\,.$$

The general and partial solutions of these equations are also expressed through the general and partial solutions of the Poisson equations and the inhomogeneous Helmholtz equations.

 ISBN 978-3-949059-50-6

Conclusions

The solution of the transformed systems of differential equilibrium equations, which are derived from the initial equilibrium equations of high orders, is reduced to the solution of homogeneous and inhomogeneous second-order differential equations.

Formulas for general and partial solutions of inhomogeneous differential equations of high order equilibrium through general and partial solutions of inhomogeneous differential equations of the second order (Poisson equations and inhomogeneous Helmholtz equations) are derived.

This greatly simplifies the solution of boundary value problems, especially in the case of intermittent or local loads.

The developed method of reduction of inhomogeneous differential equations of high orders to equations of the second order can also be used in solving boundary value problems based on other plate theories.

 ISBN 978-3-949059-50-6