KAPITEL 5 / *CHAPTER 5* [5]
MATHEMATICAL MODELLING OF MANUFACTURING PROCESSES OF PARTS AND STRUCTURAL ELEMENTS BY BUILD-UP METHODS

DOI: 10.30890/2709-2313.2022-12-01-002

Introduction

We encounter build-up processes in the study of various technological and natural processes such as winding, sputtering, deposition, freezing, accretion, as well as in the sequential erection and loading of structures and building constructions, crystal growth, phase transformations in solids [1 – 8], etc.

The theoretical substantiation of new technologies for the production of shells, pipes and other parts of rotation by the build-up [5 – 8], requires the development of calculation methods that more fully reflect the properties of the material from which this or that part is made. It is known, in particular, that polymers and composite materials [1, 3, 5, 9], which are used in the manufacture of various parts and structural elements, have pronounced creep properties. This circumstance leads to the redistribution of stresses in the parts during the build-up process, changes in shape and size after manufacturing and when loaded.

The mechanics of such processes can be studied from different points of view. One of them is a model representation of the processes of building up real structures based on the theory of creep of heterogeneously ageing bodies [10, 11] (the theory of growing bodies). They are characterized by the fact that during the build-up, different elements of these bodies are produced (originated) at different moments of time, that is, the age of the material in these bodies is not the same and depends on spatial coordinates. Therefore, along with the traditional heterogeneity, there is a specific heterogeneity due to the fact that the process of natural or artificial ageing in these bodies proceeds differently in all elements.

Depending on specific conditions, the process of building up viscoelastic bodies can occur both discretely and continuously. In the first case, elements of finite dimensions made of material of different ages are successively attached to the body at separate moments of time. In the second case, i.e. with continuous growth, an element of infinitesimal thickness is added to the body for each infinitesimally small interval of time.

Before moving on to the modelling of the building-up process, we make one

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remark concerning the general concepts of the build-up of deformable solids.

It is not difficult to imagine the process of building up a completely solid body, when the volume element added to it immediately after joining, as it were, "freezes" and loses the ability to deform in the future. In this case, the configuration of the body observed by us at any arbitrary stage of growth is "frozen", that is, it does not undergo any further changes in volume or shape. This "solidified body" simply "overgrows" with new material, as a result of which the shape and volume of the entire growing body are constantly changing.

Thus, when we talk about the changeability of the configuration and dimensions of a growing solid body, we will mean the change of these parameters not as a result of a force-deforming effect on the body, but as a result of joining it with new elements, adding material, imposing additional connections, etc.

In this work, we do not dwell in detail on the types of augmentation [1, 12] and their comparative analysis [13 – 15], but examines the problems that arise, mainly, in the manufacture of rotation parts and structural elements from polymer material using build-up methods.

It is known from the literature [10; 11] that solutions to viscoelasticity problems for growing bodies, model, mainly, the processes of manufacturing rotation parts and erecting construction structures from concrete. In these tasks, the build-up process is assumed to be carried out from the outside of the part with a given law of change of radius and external surface load.

This chapter deals with the problems of the internal and external build-up of an empty circular viscoelastic cylinder made of polymer material, which is in the conditions of a uniaxial stress state in the case of a linear creep law to simulate the process of manufacturing rotating parts by build-up methods.

5.1. Basics of the technology of build-up processes

We focus on the modelling of such build-up processes as winding and sputtering, which are used during the manufacture of thermal insulation of pipes and tanks, as well as during the manufacture of various rotating parts.

We consider several options for sputtering thermal insulation. Among the materials used for thermal insulation of tanks, polyurethane foam has become widespread. Thermal insulation from it can be block and monolithic [16]. Monolithic insulation is made by spraying on the walls of the tank with the help of special machines. Foam plants for applying thermal insulation to tanks up to 14 m high are a dosing unit consisting of two gear pumps of the НШ20ДО2 type, driven by an electric motor and a gearbox, which ensures the specified mode of operation of the pumps. Components of thermal insulation (polyester resin and diurethane diethylene glycol) are taken from the containers with separate hoses and fed to the dispenser gun. A hose for supplying compressed air is connected to it. The components in the containers are heated and continuously mixed by gear pumps of the РЗ type. The air that is sprayed is cleaned with the help of a moisture-oil separator and an oil-capturing filter. All modes of operation of the plant are carried out from the control panel, which is located at a distance from the spray gun. The spray gun, which is on a special carriage, moves vertically and horizontally. The movement mechanism has a special device for reciprocating movement.

An approximately similar design of the device for spraying polyurethane foam on tanks is proposed by the Shell Company. It consists of a guide rail, which is mounted at a distance of 80 cm from the bottom of the tank, along which the trolley moves with the help of a special motor. A vertical rack is attached to the cart, along which the spray gun, which is driven by a special motor, can move up and down. The initial components of thermal insulation are supplied by flexible hoses to the sprayer and are applied to the wall of the tank under the action of compressed air. The required thickness of thermal insulation is provided by the specified number of passes. Externally, thermal insulation can be covered with a protective cover. Sometimes, the same polyurethane foam is used as a protective cover, but in a different proportion. As a result of sputtering, the outer layer of thermal insulation turns out to be very dense and strong, well confronting external loads, penetration of moisture and exposure to ultraviolet rays.

To prevent cracking of the monolithic thermal insulation during emptying and filling of the tank (a sudden change in the temperature of the inner wall of the tank), sometimes before applying the main thermal insulation to the outer surface of the tank, a buffer insulating gasket is applied with the help of adhesives. This gasket absorbs all the thermal fluctuations of the tank walls, and the main insulating material practically does not deform.

To improve the coating (increase its homogeneity and thermophysical properties), it is necessary to thoroughly mix the components of the thermal insulation before feeding them into the spraying device. This can be provided by a

 ISBN 978-3-949059-59-9

special unit (Fig.1), consisting of cylinder *3*, inside which moves a piston *2*, which is driven by device *1*. The cylinder passes into chamber *4*, above which hopper *5* for the mixture is located. The mixture from the hopper is fed to the chamber through the inlet valve *6*. From chamber *4*, it is squeezed out through the discharge valve *7* into the mixing chamber *8*. Inside the mixing chamber, there is a splitter *10*. Compressed air is supplied to the mixing chamber through a pipe *9*. The heat-insulating mixture supplied to the mixing chamber, is pressed against its walls thanks to the splitter and well mixed with compressed air. The formed aerosol mixture moves through the hose *11* and is sprayed onto the protected object through a special device *12*.

For the insulation of tanks intended for the preservation of liquids with high temperatures or liquefied natural gases, a folded metal shell with a felt pad between the surface to be protected and the shell, is proposed. Polyurethane foam is pumped into the shell through an opening. After filling, the opening is closed. A set of these shells forms the thermal insulation of the tank. They are attached to the wall either with the help of whiskers or with the help of bandage belts.

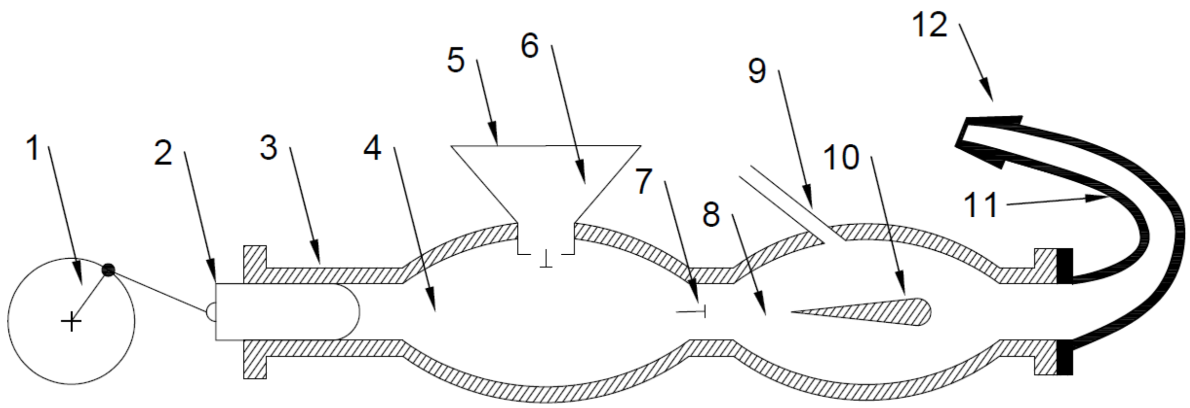

Figure 1 – The device for mixing and supplying components of polyurethane insulation

Thermal insulation of pipes by sputtering requires special conditions and plants, but this method is more technological. The specified thickness of thermal insulation can be obtained in one or more passes along the pipe of the thermal insulation plant. The same equipment can be used to insulate pipes of different diameters. The quality and thickness of the insulating coating are continuously monitored during the works. Applying the insulating coating can be done both in the factory and under semi-stationary conditions, and at the same time rhythmicity and continuity of work are ensured.

To protect the thermal insulation from mechanical damage and the penetration of moisture to it, it is protected from the outside with casings: made of tape or other materials. In complex sections of pipelines (crossings through rivers, roads, swamps, etc.), the thickness of the thermal insulation is much greater than on the main highway, and the protective casing is made thicker and completely airtight.

The principle of operation of plants for sputtering thermal insulation is as follows. Various containers contain a polyester mixture with additives for extinguishing the flame, the required amount of water, catalysts and diurethane diethylene glycol. Components from containers are fed into the mixing chamber of the spray gun by dosing pumps. Compressed air is supplied to the same chamber, which intensively mixes the components and ejects it through a hose with a nozzle onto the surface being processed. Obtaining the specified qualities of the heat-insulating coating is ensured by the necessary adjustment of the supply speed of the components and the speed of their hardening on the surface, taking into account the temperatures of the environment, components and the object being insulated. In this way, pipes are insulated both in the factory and in the field. Compared to the pouring method, polyurethane foam thermal insulation by sputtering has better uniformity in the axial and radial directions, its density decreases (up to 30 kg/m^3). This method allows you to obtain thermal insulation of any thickness due to one or more passes of the spray gun relative to the surface to be protected.

The main disadvantage of this method is the insufficient adhesion of the protective casing to the insulation material. The protective casing is most often made of the pellicle, but other materials are also used. Their connection with thermal insulation occurs with the help of various glues.

A significant number of plants for applying polyurethane foam thermal insulation by the sputtering method have been proposed. We consider some of them.

A rotary spray gun plant consists of a carriage that is moved along the pipe by a motor, and cleaning and atomization systems that rotate around the pipe. The rotation and translational movement of the cleaning and dusting systems is carried out by a gear driven by the trolley wheels. The cleaning system consists of metal brushes mounted on the inside of the drum, and the spray system includes a spray gun that also rotates evenly around the pipe. The big disadvantage of this plant is the combination of the process of cleaning the pipe and dusting the insulation since the dust generated during cleaning sharply worsens the working conditions. In addition, the plant is quite complicated due to the presence of a rotating spray gun. Heat-insulating mass and compressed air are supplied to the gun through a system of small-diameter pipelines in special drums that rotate around the pipe. The need for various gaskets and seals does not eliminate the cost of heat-insulating mass. To improve the adhesion of the heat-insulating mass to the surface to be insulated, the latter is heated to 313 K, and the heat-insulating mass to 300 K. Externally, the thermal insulation is protected by special fabrics, paper or pellicle, which are wound from coils fixed on a cart. The following parameters are recommended for the operation of the plant during the insulation of pipes with a diameter of 219 mm: the thickness of the insulating layer is 50 mm, the rotation frequency is 0.416 s^{-1}, the linear speed of the carriage is 0.05 m/s. Using a self-propelled plant, a pipeline with a diameter of 0.168 m was covered with polyurethane foam insulation with a thickness of 76 mm at a speed of 0.1 m/s. Pipe cleaning was not carried out in this plant. The entire plant is mounted on a special frame of a crawler tractor. Externally, thermal insulation was protected by a polyethylene pellicle wound from bobbins fixed on a drum. The polyethylene pellicle was wound on the surface of the thermal insulation covered with glue. The places where the pellicle was joined were also glued.

The plant with stationary spray guns is shown in Fig.2. Pipe *1* being insulated passes through the cleaning device *5*. Moving in the axial direction and rotating around its axis, the pipe passes the first isolation post (*6, 7*) where a layer is applied to the surface of the pipe adhesive *4*. It consists of a flat nozzle *7* and a worm gear *6*. The adhesive is applied to the pipe rotating in the form of a flat tape *8*. The speed of the axial movement of the pipe is selected so that the adhesive is applied with an overlap. The next insulation post applies thermoplastic insulation to the pipe. Here, a heat-insulating mass is applied to the pipe with a flat tape *11* through a flat mouthpiece *10*, fixed in a worm gear *9*. The frequency of rotation and movement of the pipe is selected so that the thickness of the thermal insulation *3* corresponds to the calculation. The pipe with insulation is covered with glue through pipe *2*. The outside of the insulated pipe is covered with polyethylene pellicle, which is placed on top of the glue. Flat mouthpieces together with reducers have the possibility of horizontal movement in the case of insulating pipes of different diameters.

The principle of rotation and axial movement of the pipe is used in various plants. In them, thermal insulation is applied to a rotating pipe, and the spray gun is moved along the axis of the pipe in both forward and reverse directions. To speed up the insulation process, some plants are provided with thermal devices for drying the insulation. These devices are mounted in such a way as not to interfere with the process of applying insulation. The applied layer of insulation is dried with thermal devices, and then another dose is applied. The speed of insulation drying is regulated by the temperature of thermal devices and the speed of movement of the insulating layer relative to the device. The operating mode is selected so that when the pipe is rotated and the nozzles are moved (in one or more passes), the specified thickness of polyurethane foam insulation is obtained. For this purpose, the plants are equipped with appropriate control and measurement equipment. They are designed to work in stationary conditions (a factory, a pipe welding base in highway conditions or a specially created production area).

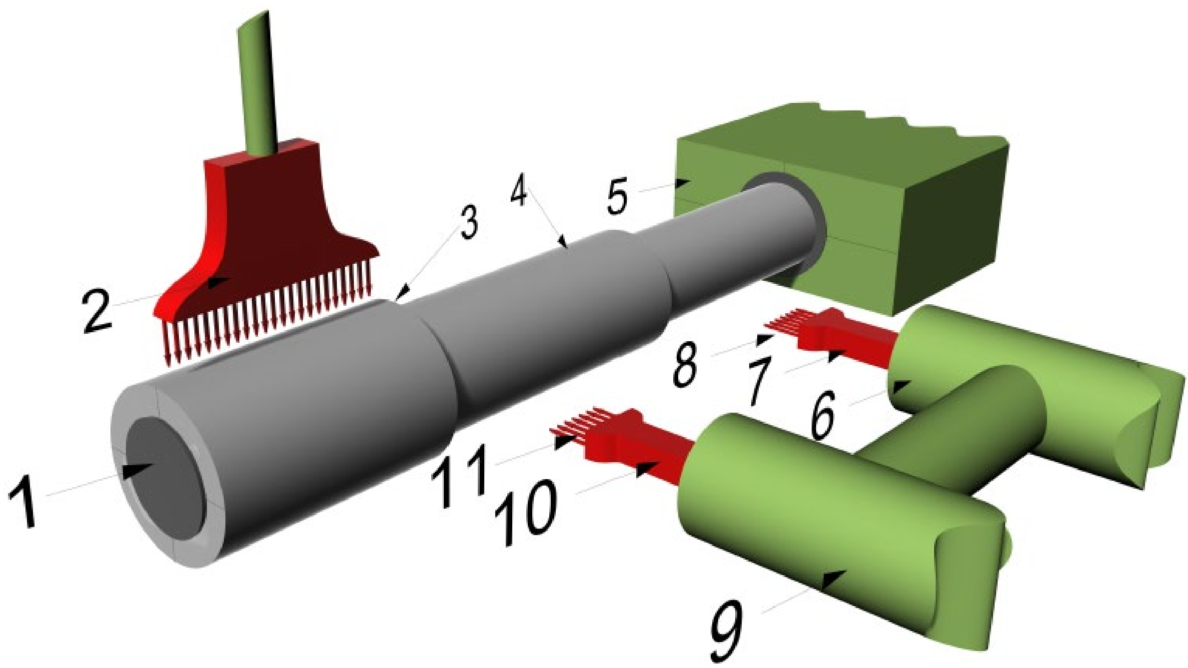

Figure 2 – Plant with fixed spray guns for applying polyurethane foam insulation

Fig.3 shows a diagram of *a plant with a mobile spray gun*, which can insulate pipes with a diameter of 500 to 1200 mm and a length of up to 12 m. The main units of the plant are a roller stand for fixing and rotating the pipe and a spraying device capable of moving along the pipe. They consist of cones *3* and *13*, on which the insulated pipe *4* is installed. Cones are fixed on supports *2* and *14* with foundations *15*, *16*. Cone *3* is driven by an electric motor with a gearbox *1*. A frame with rail *7* and sprockets *6* and *11* is attached to the foundation. A carriage *10* is installed on the rail, on which a hose *9* with a sprayer is fixed for feeding heat-insulating mass to the rotating pipe. The carriage *10* is set in motion by the chain transmission *8*, which is

ISBN 978-3-949059-59-9

powered by the electric motor *5*. The polyurethane foam component, applied to the pipe by the moving carriage, is foamed, forming a given thickness of thermal insulation *12*. The given thickness of the insulation is ensured by the appropriate selection of the rotation frequency of the pipe and the movement of the carriage in one or several passages. Sometimes, instead of a carriage, a self-propelled cart is used, on which, in addition to the spraying device, containers for insulation components, pump dosing units with control devices and controls are installed.

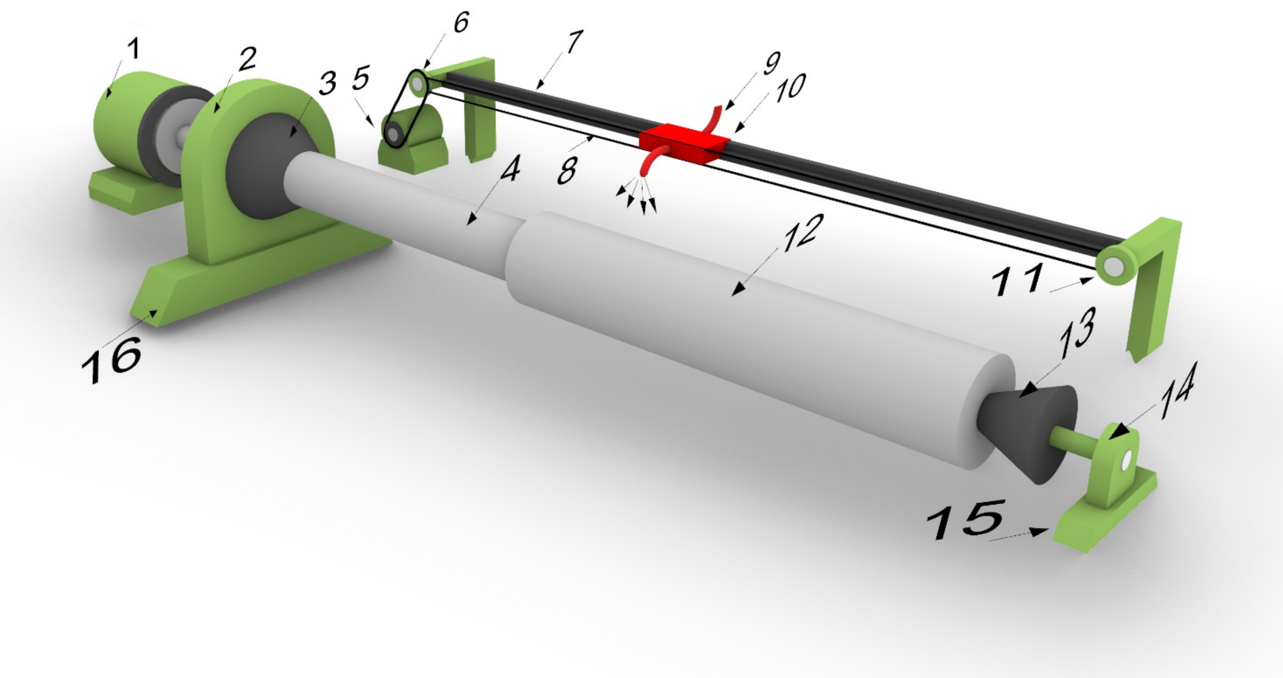

Figure 3 – Plant scheme with a mobile spray gun for applying polyurethane foam insulation

Thus, various devices can be used to apply thermal insulation to tanks and pipelines. The choice of the device should be based on the heat-insulating material and take into account the economic factors of the work.

5.2. Mathematical model of a growing linear viscoelastic body under small deformations in the case of a uniaxial stress state

The production of a viscoelastic body begins at the moment $t = 0$ on the absolute time scale. The moment of nucleation (production) of the element of this body around the point $\mathbf{x} = (x_1, x_2, x_3)$ on the absolute time scale is denoted by $\tau^*(\mathbf{x})$.

The function of spatial coordinates $\tau^*(\mathbf{x})$ determines the speed and sequence of production of a viscoelastic body or the law of assembly of a structure from viscoelastic elements. General restrictions imposed on the function $\tau^*(\mathbf{x})$ are usually boundedness and piecewise continuity requirements.

If the material, of which a viscoelastic body is made or construction is erected, has the property of ageing, and the manufacturing (construction) process is stretched over time, then the body or construction under consideration will have age heterogeneity, as a result of which its physical and mechanical properties will depend both on time and spatial coordinates.

The dependence of elastic and rheological properties on spatial coordinates in such bodies is caused by the fact that the ageing process in individual elements proceeds with one or another delay or advance, which depends on the age of a given element of the body. Such bodies are called *heterogeneously ageing*.

We can talk about age heterogeneity of a different nature when the body is exposed to a heterogeneous external field (temperature, radiation, etc.), and therefore the ageing process proceeds differently in all its elements. Then the age of the elements will depend on the characteristics of the external field, and therefore on the spatial coordinates, as in the previous case.

We consider a prismatic sample made of ageing viscoelastic material of the same age and loaded with a longitudinal force. We will assume that the sample is in a uniaxial stress state. *The total specific deformation of the sample* $\delta(t,\tau)$ until the moment of time t under the action of the axial unit load applied at the age τ, is determined by the dependence [10]

$$\delta(t,\tau) = \frac{1}{E(\tau)} + C(t,\tau), \tag{1}$$

where $E(\tau)$ is the age-dependent instantaneous modulus of elasticity of the material,

$C(t,\tau)$ is, so-called, *measure of creep*, equal to the creep deformation of the sample up to the moment of time t. Thus, the total specific deformation consists of two terms, the first of which is the instantaneous elastic deformation that occurs at the moment of application of a unit load and is equal to $\frac{1}{E(\tau)}$ and the second is the creep deformation caused by the action of the same unit load.

At the instant of time, $\tau = \tau_0$ stress is applied to the prismatic sample under consideration, which changes according to the law $\sigma_1 = \sigma_1(t)$ under $t \geq \tau_0$. Then, on the basis of relation (1) and *the principle of superposition*, the complete *relative* deformation is expressed as dependence

$$\varepsilon_1(t) = \sigma_1(t)\delta(t,\tau_0) + \int_{\tau_0}^{t} \frac{d\sigma_1(\tau)}{d\tau}\delta(t,\tau)d\tau. \tag{2}$$

Integrating by parts the second term in (2), we obtain

$$\varepsilon_1(t) = \frac{\sigma_1(t)}{E(t)} - \int_{\tau_0}^{t} \sigma_1(\tau)\frac{d}{d\tau}\delta(t,\tau)d\tau. \tag{3}$$

This ratio is a rheological equation of the creep theory of a homogeneously ageing medium under uniaxial stress. It can be seen from equation (3) that the total longitudinal strain $\varepsilon_1(t)$ of elastic moment strain (first term) and creep strain (second term).

It can be shown that there is an inequality in the interval $\tau_0 \le \tau \le \infty$

$$\frac{\partial}{\partial\tau}\delta(t,\tau) = \frac{\partial}{\partial\tau}\left[\frac{1}{E(\tau)} + C(t,\tau)\right] \le 0,$$

from which it follows that the creep strain is always integral at $\sigma_1 \ge 0$.

In some cases, it is advisable to write the rheological equation (3) in the form

$$\varepsilon_1(t) = \frac{\sigma_1(t)}{E(t)} - \int_{\tau_0}^{t} \frac{\sigma_1(\tau)}{E(\tau)}\left[E(\tau)\frac{d}{d\tau}\delta(t,\tau)\right]d\tau.$$

Now we consider a prismatic sample (rod) that is inhomogeneous in time, made of an ageing material and which is in a uniaxial stress state. The heterogeneity of the rod is that the age of the rod material is a function of the longitudinal coordinate x_1. The moment of material nucleation in the vicinity of the point with the coordinate x_1, in the absolute time scale, will be denoted by $\tau^*(x_1)$.

We select an elementary segment of the rod and introduce a local time scale t', for this element, considering that the zero moment of time $(t' = 0)$ corresponds to the moment of material nucleation in this element. We write the defining equation for a locally homogeneous rod element in the form

$$\varepsilon_1'(t) = \frac{\sigma_1'(t')}{E(t')} - \int_{0}^{t'} \frac{\sigma_1'(\tau')}{E(\tau')}Q(t',\tau')d\tau', \tag{4}$$

where ε_1', σ_1' – axial strain and stress at the point x_1, $Q(t',\tau')$ – *core of material creep*. Dashes mean, as already noted, that the beginning of time (local time zero) for the element of the rod under consideration coincides with the moment of its origin in the absolute time scale.

We replace the variable in (4): $t' = t - \tau^*(x_1)\left(\tau' = \tau - \tau^*(x_1)\right)$, where t is time of the absolute scale, the same for all elements. We get

$$\varepsilon_1(t) = \frac{\sigma_1(t)}{E\left(t - \tau^*(x_1)\right)} - \int_{\tau_0}^{t} \frac{\sigma_1(\tau)}{E\left(\tau - \tau^*(x_1)\right)} Q\left(t - \tau^*(x_1), \tau - \tau^*(x_1)\right) d\tau, \qquad (5)$$

where ε_1, σ_1 are deformation and stress corresponding to the moment t in an absolute time scale, a $\tau_0 = \tau_0(x_1)$ is the moment of applying forces to a body element around a point with a coordinate x_1, in the same timeline.

For each x_1 and given strain $\varepsilon_1(t)$, relation (5) is the Volterra integral equation of the second kind with respect to the stress $\sigma_1(t)$. An overview of works devoted to these equations can be found, for example, in [7, 11]. In particular, if at a fixed x_1, the core $Q\left(t - \tau^*(x_1), \tau - \tau^*(x_1)\right)$ and the functions $\varepsilon_1(t)$, $E\left(t - \tau^*(x_1)\right)$ are integrated in quadratures at $\tau_0(x_1) \le t \le T$, then equation (5) has a unique solution $\sigma_1(t)$ in the interval $\tau_0(x_1) \le t \le T$, which is expressed in elementary functions.

Ratio (5) determines the equation of state of the linear theory of creep (viscoelasticity) for heterogeneously ageing bodies under uniaxial stress in the case of small deformations.

Due to the fact that the creep cores in these equations clearly depend on the spatial coordinate, nonuniformly ageing viscoelastic bodies will sometimes be called *nonuniformly viscoelastic* bodies.

The main aspect of problems of solid body creep is the need to take into account the time factor when determining the stress-strain state.

This statement of the problem assumes that the creep characteristics of the material of the given body are known – some functions of time and constants determined from the experiment [10, 17]:

a) instantaneous deformation modulus;

b) measure of creep for a uniaxial stress state;

c) measure of creep in pure shear;

d) coefficient of transverse compression for elastic deformation,

as well as some parameters included in these functions.

It is very difficult to obtain analytical expressions for these functions in the general case of a spatial stress state. Therefore, when constructing the theory of creep, it is necessary to proceed directly from experimental creep curves. At the same time, the structure of analytical expressions for the above-mentioned characteristics should reflect the main properties of the creep process in real bodies and not contradict the results of experimental data. At the same time, the theory of creep should be such that it can be counted on to solve its main problems by the usual means of mathematical analysis.

The idea of describing phenomena such as aftereffect and creep using integral equations with a variable upper limit belongs to Boltzmann. It was further developed by Volterra. In most works devoted to the continuous theory of elasticity, the cores of integral equations are assumed to depend on the difference between two arguments – the time of application of the load τ and the moment of observation t. This condition follows the requirement to implement a closed Volterra cycle, which expresses the invariance of the integral relation with respect to the change in the start of the time countdown.

It is known [7; 11] that integral equations with such cores reduce to linear equations with constant coefficients and the theory of heredity is interpreted as the Maxwell-Thomson theory of an elastic-viscous or elastic-relaxing body.

When choosing an analytical expression of the creep rate $C(t,\tau)$ for some materials that have this property, it is necessary to take into account the basic conditions that this expression must satisfy.

Numerical experiments [10, 17 – 19] show that

a) $C(t,\tau) > 0$ for all values $t > \tau$; $C(t,\tau) = 0$ under $t = \tau$;

b) $\lim\limits_{t\to\infty} \frac{\partial C(t,\tau)}{\partial t} = 0$ for all values $0 < \tau \leq t$;

c) the creep function $C(t,\tau)$ should monotonically decrease depending on the age of the material $\frac{\partial C(t,\tau)}{\partial \tau} < 0$, and so that $\lim\limits_{\tau\to\infty} C(t,\tau) = C_0$, where C_0 is the limiting value of the creep rate for this material;

d) starting from a certain age $\tau = \tau_0$, the value of the creep function $C(t,\tau)$ should differ arbitrarily little from the creep function for old or already ageing material.

Based on these understandings, the expression for the creep rate $C(t,\tau)$ is accepted

$$C(t,\tau) = \varphi(\tau) f(t-\tau). \quad (6)$$

The sum of exponential functions is used to approximate the function $f(t-\tau)$

$$f(t-\tau)=\sum_{k=0}^{m}B_k e^{-\gamma_k(t-\tau)}, \tag{7}$$

where B_k and γ_k – constants that are selected appropriately for this material, moreover $B_0=1,\ \ \gamma_0=0$ i $\gamma_k>0\ (k=1,2,3,\dots,m)$, a $\varphi(\tau)$ – while the arbitrary monotonically decreasing function meets the condition

$$\lim_{\tau\to\infty}\varphi(\tau)=C_0. \tag{8}$$

Then the relation (6) for $C(t,\tau)$ finally takes

$$C(t,\tau)=\varphi(\tau)\sum_{k=0}^{m}B_k e^{-\gamma_k(t-\tau)}. \tag{9}$$

This representation of the creep rate $C(t,\tau)$ is characterized in that it reflects the main properties of the phenomenon of material creep over time, namely, its ageing and continuity.

Due to condition (8), expression (9), starting from some age $\tau\geq\tau_0$, will satisfy the asymptotic equality

$$C(t,\tau)\approx C_0\sum_{k=0}^{m}B_k e^{-\gamma_k(t-\tau)},$$

which coincides quite well with the experimental creep curves for a number of materials, at their old age, and the difference in the properties of these materials will be determined by the proper selection of parameters $C_0,\ B_k$ i $\gamma_k\ (k=1,2,3,\dots,m)$.

It follows from this that the speed of the function $\varphi(\tau)$ going to a constant value C_0 characterizes the moment when the old age of this material is approaching. In addition, the form of the function $\varphi(\tau)$ should be chosen so that for the age of the material $0<\tau\leq t$, the expression of the function $C(t,\tau)$ agrees well with the experimental creep curves of this material.

Based on the above considerations and as a result of processing a number of experimental data [17] the function $\varphi(\tau)$ in the general case is represented by the expression

$$\varphi(\tau)=C_0+\sum_{k=1}^{m}\frac{A_k}{\tau^k}\quad(\tau>0), \tag{10}$$

where C_0 is the limit value of creep rate for this material, A_k – some parameters depending on the properties and aging conditions of the material. The values A_k

should be selected so that equation (9) best describes the experimental creep curves of this material.

In [10, 11, 17] some experimental data related to the creep of various materials, including metals exposed to high temperatures, are presented. When analysing and evaluating them, it turns out that in relations (7) and (10) it is enough to limit ourselves to two members of the series, that is, to accept

$$\varphi(\tau) = C_0 + \frac{A_1}{\tau}, \;\; f(t-\tau) = 1 + B_1 e^{-\gamma_1(t-\tau)}.$$

The numerical values A_1, B_1, C_0 and $\gamma_1 = \gamma$ are selected so as to obtain maximum agreement with the experimental creep curves of the required material. From the same sources, the specified parameters will be selected when solving the problems discussed below.

Thus, if the law of change of creep rate is represented in the general case by the expression

$$C(t,\tau) = \varphi(\tau)\left[1 - e^{-\gamma(t-\tau)}\right], \tag{11}$$

where

$$\varphi(\tau) = C_0 + \frac{A_1}{\tau},$$

then it gives a fairly close coincidence with the experimental curves of the creep of metals (at elevated temperatures) both in their old and young ages, and also reflects the main properties of the phenomenon of material creep over time, namely its ageing and heredity.

It is possible that in some cases, for a better match with the experimental creep curves of the material under study, it will be necessary to preserve not two, but more terms of the series in expressions (7) and (10), or even choose a different form of dependence for $\varphi(\tau)$.

Therefore, in the future, when solving the main equations of the theory of viscoelasticity, in order to preserve commonality, we will assume that the measure of creep $C(t,\tau)$ is expressed by a dependence of the general form (11), in which the function $\varphi(\tau)$ should only be monotonic and satisfy the condition (8).

In addition to the creep rate, a few words should be said about one more characteristic – the elastic moment deformation modulus $E(\tau)$.

Numerical studies show [11, 17], that the modulus of elastic instantaneous deformation of ageing materials with increasing age τ grows, approaching the limit value of the modulus of elasticity E_0 for curves of very old age. At the same time, it follows from the experimental curves of elastic moment deformations that in the case of natural ageing, the function $E(\tau)$ is a continuous and limited function of the age τ over the entire interval of its change $\tau^* \leq \tau < \infty$, τ^* – is the time of manufacturing (nucleation) of the material. Summarizing the available experimental data, it can be assumed that the function $E(\tau)$ satisfies the following conditions:

$$E(\tau) > 0, \quad \lim_{\tau\to\infty} E(\tau) = E_0,$$

$$\frac{dE(\tau)}{d\tau} \geq 0, \quad \lim_{\tau\to\infty} \frac{dE(\tau)}{d\tau} = 0 \quad \forall\tau \geq \tau^*.$$

In the future, we will use the form of the dependence of the instantaneous modulus of elasticity of the ageing material on age $E(\tau) = E_0\left(1 - A\, e^{-\beta\,\tau}\right)$, where $A,\ \beta$ are positive constants. This type of dependence is experimentally confirmed in [10, 11, 17].

The values of the creep parameters adopted in the calculations are given when solving specific problems.

5.3. The method of calculating the stress-strain state of viscoelastic growing bodies during the build-up process

We consider an empty homogeneous viscoelastic circular cylinder, the cross-section of which is a circular ring of inner radius a_0 and outer radius b_0 (Fig.4). Starting from the moment of time $t = 0$, an internal pressure $P(t)$, $P(0) = P_0$ is applied to the cylinder from the inside and its continuous increase with a homogeneous viscoelastic material begins, so that the inner radius of the cylinder decreases monotonically according to the law $a(t)$, $a(0) = a_0$. At the moment of time T, the inner radius of the cylinder reaches the value a_1, and the growth process stops, i.e. $a(t) = a_1$ at $t \geq T$.

The functions $a(t)$ and $P(t)$ are assumed to be continuously differentiable on the interval $0 < t < T$. It is necessary to determine the stress-strain state in the cylinder for all $t \geq 0$.

Solving the given problem, we use the approach outlined in [7, 9, 20, 21].

We consider the plane deformation of the cylinder, that is, we set $u_z = 0$. Here, the z-axis is directed along the axis of the cylinder, and the symbol z denotes the displacement of the points of the cylinder along the z-axis.

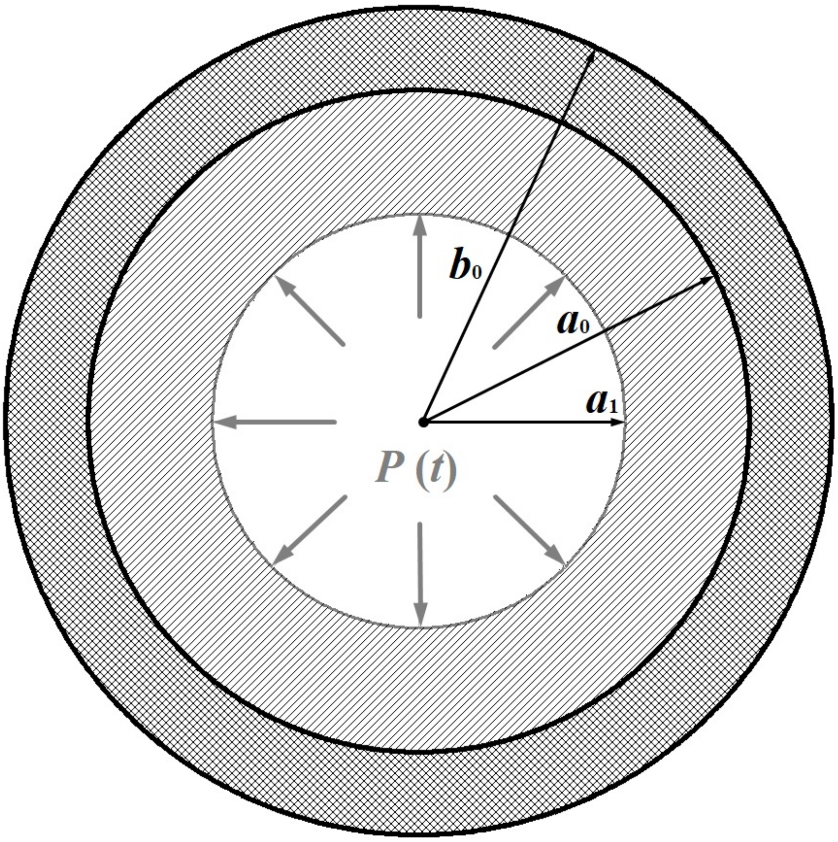

Figure 4 – The scheme of building up a viscoelastic hollow cylinder

We note down the main equations of the problem:

condition of incompressibility –

$$\varepsilon_r + \varepsilon_\vartheta = 0, \tag{12}$$

equilibrium equation –

$$\frac{\partial\, \sigma_r}{\partial\, r} = -\frac{\sigma_r - \sigma_\vartheta}{r}, \tag{13}$$

Cauchy relation for rates of deformations and displacements –

$$\dot{\varepsilon}_r = \frac{\partial\, \dot{u}_r}{\partial\, r}, \quad \dot{\varepsilon}_\vartheta = \frac{\dot{u}}{r}. \tag{14}$$

Here are r, ϑ – polar coordinates in the plane of the cross-section of the cylinder, $\varepsilon_r = \varepsilon_r(t, r)$ and $\varepsilon_\vartheta = \varepsilon_\vartheta(t, r)$ – deformation components, the dot denotes the time derivative.

ISBN 978-3-949059-59-9

Since all deformation components, except for ε_r i ε_ϑ, are equal to zero, taking into account (12), we have $\varepsilon_u = (2\varepsilon_{ij}\varepsilon_{ij})^{1/2} = 2|\varepsilon_r| = 2|\varepsilon_\vartheta|$ for the intensity of deformations. From here and from the defining equations of the nonlinear theory of creep for heterogeneously ageing bodies, due to the condition $\varepsilon_\vartheta > 0$, the equation of state can be written in the form

$$\sigma_r(t,r) - \sigma_\vartheta(t,r) = 2G_1(t-\tau^*(r))\,(\varepsilon_r(t,r) - \varepsilon_\vartheta(t,r))\varepsilon_\vartheta^{m-1}(t,r) -$$

$$- \int\limits_{\tau^*(r)}^{t} R_1\big(t-\tau^*(r), \tau-\tau^*(r)\big)\big(\varepsilon_r(\tau,r) - \varepsilon_\vartheta(\tau,r)\big)\varepsilon_\vartheta^{m-1}(\tau,r)d\tau, \qquad (15)$$

where $G_1 = G \cdot 2^{m-1}$; $R_1 = R \cdot 2^{m-1}$; $m = 1,2,3,\ldots$ – parameter characterizing the degree of nonlinearity of the creep law; G – elastic moment modulus of the material; $R = R(t,\tau)$ – relaxation core of viscoelastic material; $\tau^*(r)$ – the moment of nucleation of the elementary layer of the cylinder, τ– the age of the material in which the load is applied to it.

The function $\tau^* = \tau^*(r)$ is equal to zero at $a_0 \le r \le b_0$and coincides with the inverse function $a(t)$ at $a_1 \le r \le a_0$, i.e. $\tau^*\big(a(t)\big) \equiv t$, $a\big(\tau^*(r)\big) \equiv r$, $\tau^*(a_0) = 0$, $a(0) = a_0$.

The boundary conditions have the form

$$\sigma_r(t,b_0) = 0, \quad \sigma_r\big(t,a(t)\big) = -P(t), \quad 0 \le t < T;$$

$$\sigma_\alpha(t,a_1) = 0, \quad t \ge T, \quad \alpha = r,\vartheta. \qquad (16)$$

Differentiating (12) with respect to time and substituting (14) into the resulting relation, we come to the equation

$$\frac{\partial\,\dot{u}_r}{\partial\,r} + \frac{\dot{u}_r}{r} = 0.$$

The last relation is nothing but a differential equation of the first order with respect to the function $\dot{u}_r = \dot{u}_r(t,r)$. Solving this equation, taking into account (14), we obtain

$$\dot{u}_r = \frac{c(t)}{r}, \qquad \dot{\varepsilon}_r = -\dot{\varepsilon}_\vartheta = -\frac{c(t)}{r^2}, \qquad (17)$$

where $c(t)$ – some function to be defined.

At the moment of instantaneous application of load $P(t)$ (that is, at $t = 0$), the velocities of deformation components ε_r, ε_ϑ and displacement u_r contain singular components of the form $\Delta\varepsilon_r(r)\delta(t)$, $\Delta\varepsilon_\vartheta(r)\delta(t)$, $\Delta u_r(r)\delta(t)$, where $\Delta\varepsilon_r$, $\Delta\varepsilon_\vartheta$, Δu_r

are the increments of the corresponding values at the moment $t = 0$, and $\delta(t)$ is the Dirac delta function. Therefore, at $t = 0$, the Cauchy relations are fulfilled for the increments of deformations and displacements. Using the above considerations, it can be shown that the solution obtained below is also valid for an arbitrary piecewise continuous load $P(t)$.

Taking into account the initial condition $u_r(\tau^*(r),r) = \varepsilon_\vartheta(\tau^*(r),r) = 0$ it follows from (17) that

$$u_r(t,r) = \frac{A(t) - A(\tau^*(r))}{r}, \quad (18)$$

$$-\varepsilon_r(t,r) = \varepsilon_\vartheta(t,r) = \frac{A(t) - A(\tau^*(r))}{r^2} \quad \text{under} \quad a(t) \le r < a_0; \quad (19)$$

$$u_r(t,r) = \frac{A(t)}{r}, \quad (20)$$

$$-\varepsilon_r(t,r) = \varepsilon_\vartheta(t,r) = \frac{A(t)}{r^2} \quad \text{under} \quad a_0 \le r \le b_0; \quad (21)$$

$$A(t) = -\int_0^t c(\tau)d\tau. \quad (22)$$

The integration in (22) includes the point $t = 0$. Therefore, if the singular component of the function $c(t)$ is denoted by $c_0\delta(t)$, then the boundary of the function $A(t)$ at the point $t = 0$ on the right is equal to c_0.

We express the stress $\sigma_r$through the function $A(t)$. For this, we will consider two areas.

The area $a_0 \le r \le b_0$ is the output cylinder.

Substituting the expression for the deformation components (21) into (15), we obtain (taking into account that $\tau^* = 0$, since there is no build-up process)

$$\sigma_r(t,r) - \sigma_\vartheta(t,r) = -\frac{2}{r^{2m}}\left[2G_1(t)A^m(t) - \int_0^t R_1(t,\tau)A^m(\tau)d\tau\right]. \quad (23)$$

Integrating equation (13) in the range from r to b_0, we have, taking into account the boundary conditions (16)

$$\sigma_r(t,r) = \int_r^{b_0} \frac{\sigma_r - \sigma_\vartheta}{r} dr. \quad (24)$$

Substituting (23) into (24), we finally have

$$\sigma_r(t,r) = -\left[2G_1(t)A^m(t) - \int_0^t R_1(t,\tau)A^m(\tau)d\tau\right]\int_r^{b_0}\frac{2dr}{r^{2m+1}}. \qquad (25)$$

The area $a(t) \le r < a_0$ is the growth zone.

Substituting (19) into the equation of state (15), we have

$$\sigma_r(t,r) - \sigma_\vartheta(t,r) = -\frac{2}{r^{2m}}[2G_1(t-\tau^*(r))\,(A(t) - A(\tau^*(r)))^m -$$

$$-\int_{\tau^*(r)}^{t} R_1\big(t-\tau^*(r), \tau-\tau^*(r)\big)\left(A(\tau) - A\big(\tau^*(r)\big)\right)^m d\tau]. \qquad (26)$$

Integrating (13) in the range from $a(t)$ to r, taking into account the boundary conditions (16), we obtain

$$\sigma_r(t,r) = -P(t) - \int_{a(t)}^{r}\frac{\sigma_r - \sigma_\vartheta}{r}dr. \qquad (27)$$

Substituting (26) to (27), we have

$$\sigma_r(t,r) = -P(t) + \int_{a(t)}^{r}\frac{2dr}{r^{2m+1}}[2G_1(t-\tau^*(r))(A(t) - A(\tau^*(r)))^m -$$

$$-\int_{\tau^*(r)}^{t} R_1\big(t-\tau^*(r), \tau-\tau^*(r)\big)\left(A(\tau) - A\big(\tau^*(r)\big)\right)^m d\tau]. \qquad (28)$$

The equation for determining the function $A(t)$ follows from the condition of continuity of stress $\sigma_r(t,r)$ at the interface of the two areas under consideration, i.e. at $r = a_0$. Substituting the value $r = a_0$ into (25) and (28) and equating their right-hand sides, we have

$$-\left[2G_1(t)A^m(t) - \int_0^t R_1(t,\tau)A^m(\tau)d\tau\right]\int_{a_0}^{b_0}\frac{2dr}{r^{2m+1}} = -P(t) +$$

$$+\int_{a(t)}^{a_0}\frac{2dr}{r^{2m+1}}[2G_1(t-\tau^*(r))(A(t) - A(\tau^*(r)))^m -$$

$$-\int_0^t R_1\big(t-\tau^*(r), \tau-\tau^*(r)\big)\left(A(\tau) - A\big(\tau^*(r)\big)\right)^m d\tau].$$

Transferring all terms to one side of the equation, and the function $P(t)$ to the other, we obtain

$$\left[2G_1(t)A^m(t)-\int_0^t R_1(t,\tau)A^m(\tau)d\tau\right]\int_{a_0}^{b_0}\frac{2dr}{r^{2m+1}}+$$

$$+\int_{a(t)}^{a_0}\frac{2}{r^{2m+1}}(2G_1(t-\tau^*(r))(A(t)-A(\tau^*(r)))^m)dr-$$

$$-\int_{a(t)}^{a_0}\frac{2dr}{r^{2m+1}}\int_0^t R_1(t-\tau^*(r),\tau-\tau^*(r))\left(A(\tau)-A(\tau^*(r))\right)^m d\tau=P(t). \quad (29)$$

In the last expression, in the second and third terms, we introduce the substitution of variables $\tau=\tau^*(r)$ $(r=a(\tau))$, then $dr=\dot{a}(\tau)d\tau$, and under $r=a_0$, $\tau=\tau^*(a_0)=0$; under $r=a(t)$, $\tau=\tau^*(a(t))\equiv t$. To prevent overlap in the notation in the last term of expression (29) (τ is a parameter and τ is a new variable), which arises in connection with this substitution, for the variable τ we enter $s=\tau^*(r)$ $(r=a(s))$, then the expression (29) are rewritten in the form

$$\left[2G_1(t)A^m(t)-\int_0^t R_1(t,\tau)A^m(\tau)d\tau\right]\int_{a_0}^{b_0}\frac{2dr}{r^{2m+1}}+$$

$$+\int_0^t\frac{2\dot{a}(\tau)}{a^{2m+1}(\tau)}\left[2G_1(t-\tau)\left(A(t)-A(\tau)\right)^m\right]d\tau+$$

$$+\int_0^t\int_0^\tau\frac{2\dot{a}(s)}{a^{2m+1}(s)}R_1(t-s,\tau-s)\,(A(\tau)-A(s))^m ds\,d\tau=P(t). \quad (30)$$

We note here

$$H_1=2\int_{a_0}^{b_0}\frac{dr}{r^{2m+1}},\quad H_2=\frac{4\dot{a}(\tau)G_1(t-\tau)}{a^{2m+1}(\tau)},$$

$$H_3(t,\tau,s)=\frac{2\dot{a}(s)}{a^{2m+1}(s)}R_1(t-s,\tau-s),$$

the equation (30) takes the form

$$H_1\left(2G_1(t)A^m(t)-\int_0^t R_1(t,\tau)A^m(\tau)d\tau\right)-\int_0^t H_2(t,\tau)(A(t)-A(\tau))^m d\tau+$$

$$+\int_0^t\int_0^\tau H_3(t,\tau,s)(A(\tau)-A(s))^m dsd\tau=P(t). \quad (31)$$

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Having determined the function $A(t)$ from the nonlinear integral equation (31), displacement $u_r$and deformation components $\varepsilon_r$and $\varepsilon_\vartheta$according to formulas (18) – (21), and stresses σ_r and $\sigma_\vartheta$according to formulas (23), (25), (26), (28). For the stress σ_z, from the condition $\varepsilon_z = 0$ and the equation of state (15), we $\sigma_z = \frac{1}{2}(\sigma_r + \sigma_\vartheta)$.

5.4. Mathematical model of the build-up process of a viscoelastic hollow cylinder under the action of internal pressure

We consider the build-up in the case of a linear creep law. This case can be investigated by putting $m = 1$ in the established relations. At the same time, from formulas (20), (21), (23), (25) we have

$$u_r(t,r) = \frac{A(t)}{r}, \quad -\varepsilon_r(t,r) = \varepsilon_\theta(t,r) = \frac{A(t)}{r^2},$$

$$\sigma_r(t,r) - \sigma_\vartheta(t,r) = -\frac{2}{r^2}\left(2G(t)A(t) - \int_0^t R(t,\tau)A(\tau)d\tau\right),$$

$$\sigma_r(t,r) = -\left[2G(t)A(t) - \int_0^t R(t,\tau)A(\tau)d\tau\right]\int_r^{b_0} \frac{2dr}{r^3} \quad \text{under} \quad a_0 \le r \le b_0; \qquad (32)$$

from (18), (19), (26), (28) we obtain

$$u_r(t,r) = \frac{A(t) - A(\tau^*(r))}{r},$$

$$-\varepsilon_r(t,r) = \varepsilon_\vartheta(t,r) = \frac{\left(A(t) - A\left(\tau^*(r)\right)\right)}{r^2},$$

$$\sigma_r(t,r) - \sigma_\vartheta(t,r) = -\frac{2}{r^2}\Big[2G(t-\tau^*(r))(A(t) - A(\tau^*(r))) -$$

$$- \int_{\tau*(r)}^t R(t-\tau^*(r), \tau-\tau^*(r))(A(\tau) - A(\tau^*(r)))d\tau\Big],$$

$$\sigma_r(t,r) = -P(t) + \int_{a(t)}^r \frac{2dr}{r^3}[2G(t-\tau^*(r))(A(t) - A(\tau^*(r))) -$$

$$- \int_{\tau*(r)}^t R\left(t-\tau^*(r), \tau-\tau^*(r)\right)\left(A(\tau) - A\left(\tau^*(r)\right)\right)d\tau] \quad \text{при} \quad a(t) \le r < a_0. \qquad (33)$$

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Next, as in the case of the general creep law, to determine the function $A(t)$ we formulate an equation that follows from the condition of continuity of stress $\sigma_r(t, r)$ at the interface of two areas, i.e. at $r = a_0$. After transformations similar to those carried out when obtaining formula (29), we have

$$A(t)\int_{a(t)}^{b_0} 2G\big(t-\tau^*(r)\big)\frac{2dr}{r^3} - \int_{a(t)}^{b_0}\frac{2dr}{r^3}\int_0^t R\big(t-\tau^*(r), \tau-\tau^*(r)\big)A(\tau)d\tau -$$

$$-\int_{a(t)}^{a_0} 2G\big(t-\tau^*(r)\big)A\big(\tau^*(r)\big)\frac{2dr}{r^3} +$$

$$+\int_{a(t)}^{a_0}\frac{2dr}{r^3}\int_0^t R\big(t-\tau^*(r), \tau-\tau^*(r)\big)\,A\big(\tau^*(r)\big)d\tau = P(t).$$

In this expression, we apply the same change of variable to the last two terms as in formula (29) and at the same time change the order of integration so that the integral

$$\int_0^t A(\tau)d\tau$$

is "external", after which we have

$$A(t)\int_{a(t)}^{b_0} 2G\big(t-\tau^*(r)\big)\frac{2dr}{r^3} - \int_0^t\Bigg[\int_{a(t)}^{b_0} R\big(t-\tau^*(r), \tau-\tau^*(r)\big)\frac{2dr}{r^3} +$$

$$+\frac{2\dot{a}(\tau)}{a^3(\tau)}\left(\int_\tau^t R(t-\tau, s-\tau)ds - 2G(t-\tau)\right)\Bigg]A(\tau)d\tau = P(t). \qquad (34)$$

Denoting in the equation (34)

$$D_1(t) = \int_{a(t)}^{b_0} 2G\big(t-\tau^*(r)\big)\frac{2dr}{r^3},$$

$$D_2(t,\tau) = \int_{a(\tau)}^{b_0} R\big(t-\tau^*(r), \tau-\tau^*(r)\big)\frac{2dr}{r^3} +$$

$$+\frac{2\dot{a}(\tau)}{a^3(\tau)}\left(\int_\tau^t R(t-\tau, s-\tau)ds - 2G(t-\tau)\right),$$

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we obtain Volterra's integral equation of the second kind for determining the function $A(t)$in the case of a linear creep law

$$D_1(t)A(t) - \int_0^t D_2(t,\tau)A(\tau)d\tau = P(t). \tag{35}$$

As it is known [7, 11, 18] the solution (35) can be obtained in quadratures if the relaxation core is taken in the form

$$R(t,\tau) = \frac{\partial\mu(t,\tau)}{\partial\tau}, \tag{36}$$

$$\mu(t,\tau) = 2G(\tau) - \varphi(\tau)\left(1 - e^{-\gamma(t-\tau)}\right). \tag{37}$$

The function $\frac{\mu(t,\tau)}{2}$ is equal to the tangential stress in a homogeneous body at the moment of time t when a unit shear deformation is applied to the body at the age τ; $\varphi(\tau)$ is an ageing function, γ is a constant, a material parameter.

Equation (35), using relations (36), (37), can be written in the form

$$A(t)\mu_1(t,t) - \int_0^t \frac{\partial\,\mu_1(t,\tau)}{\partial\,\tau}A(\tau)d\tau = P(t), \tag{38}$$

where

$$\mu_1(t,\tau) = \int_{a(t)}^{b_0} \mu\left(t - \tau^*(r), \tau - \tau^*(r)\right)\frac{2dr}{r^3}. \tag{39}$$

Substituting (37) into (39), we get

$$\mu_1(t,\tau) = 2G_2(\tau) - \varphi_2(\tau)(1 - e^{-\gamma(t-\tau)}),$$

$$G_2(\tau) = \int_{a(t)}^{b_0} G\left(\tau - \tau^*(r)\right)\frac{2dr}{r^3}, \quad \varphi_2(\tau) = \int_{a(t)}^{b_0} \varphi\left(\tau - \tau^*(r)\right)\frac{2dr}{r^3}. \tag{40}$$

Equation (38) is solved by reducing it to a second-order differential equation with respect to the function $A(t)$. Indeed, equation (38), integrating parts taking into account (39), and (40), can be written in the form

$$\int_0^t \dot{A}(\tau)\left[2G_2(\tau) - \varphi_2(\tau)\left(1 - e^{-\gamma(t-\tau)}\right)\right]d\tau = P(t).$$

Differentiating the last ratio twice with respect to t, we find

$$2G_2(t)\dot{A}(t) - \int_0^t \dot{A}(\tau)\,\varphi_2(\tau)\,\gamma\,e^{-\gamma\,(t-\tau)}d\tau = \dot{P}(t), \tag{41}$$

 ISBN 978-3-949059-59-9

$$2G_2(t)\ddot{A}(t) + 2\dot{G}_2(t)\dot{A}(t) - \gamma\,\varphi_2(t)\dot{A}(t) +$$
$$+\gamma^2\int_0^t \dot{A}(\tau)\varphi_2(\tau)e^{-\gamma(t-\tau)}d\tau = \ddot{P}(t). \qquad (42)$$

Excluding the integral from relations (41), (42)

$$\int_0^t \dot{A}(\tau)\,\varphi_2(\tau)\,e^{-\gamma\,(t-\tau)}d\tau,$$

we obtain the differential equation for determining the function $A(t)$

$$2G_2(t)\ddot{A}(t) + \dot{A}(t)\left[2\dot{G}_2(t) - \gamma\,\varphi_2(t) + \gamma 2G_2(t)\right] = \ddot{P}(t) + \gamma\dot{P}(t), \qquad (43)$$

with the following initial conditions that follow directly from (38) and (41) taking into account (39), (40)

$$A(0) = \frac{P(0)}{2G_2(0)}, \qquad \dot{A}(0) = \frac{\dot{P}(0)}{2G_2(0)}. \qquad (44)$$

Solving equation (43) with the initial conditions (44), we finally obtain the solution for the function $A(t)$

$$A(t) = \frac{P_0}{2G_2(0)} + \frac{\dot{P}(0)}{2G_2(0)}\int_0^t e^{-\eta(\tau)}d\tau + \int_0^t e^{-\eta(\tau)}d\tau\int_0^\tau \frac{\ddot{P}(x)+\gamma\dot{P}(x)}{2G_2(x)}e^{\eta(x)}dx, \quad (45)$$

where

$$\eta(\tau) = \int_0^\tau \left[\gamma + \left(2\dot{G}_2(z) - \gamma\,\varphi_2(z)\right)\,2G_2^{-1}(z)\right]dz.$$

Having determined the function $A(t)$ from relation (45), we find displacement u_r, components of deformations $\varepsilon_r, \varepsilon_\vartheta$, as well as stresses according to formulas (32), (33).

As an example, we consider the problem of build-up the above-mentioned cylinder made of polymer material that works in aggressive environments.

We take the function $\mu(t,\tau)$ in the form (37), provided that

$$G = const, \;\; \varphi(\tau) = 2G\left(C_0 + A_0 e^{-\beta\,\tau}\right), \qquad (46)$$

where C_0, A_0, β are material constants [10; 11].

The inner radius $a(t)$ of the cylinder vary according to the ratio

$$\frac{1}{a^2(t)} = \frac{1}{a_0^2} + \left(\frac{1}{a_1^2} - \frac{1}{a_0^2}\right)\frac{t}{T}, \qquad 0 \le t \le T, \qquad (47)$$

as shown in Fig.5.

Before starting directly to determine the stress-strain state of the cylinder, it is necessary to calculate the functions $G_2(\tau)$ and $\varphi_2(\tau)$, which are included in the expression for determining the function $A(t)$. For this, we use formulas (40). Parameters C_0, A_0, β, γ and values a_0, a_1 are fixed and equal to: $C_0 = 0{,}38$; $A_0 = 0{,}55$; $\beta = 0{,}015$ hour^{-1}; $\gamma = 0{,}1$ hour^{-1}; $a_0 = 0{,}9b_0$; $a_1 = 0{,}5b_0$, and the internal pressure changes according to the linear law $P(t) = P_0 - P_0 t/(2T)$. Since the stresses do not depend on the value of the elastic moment modulus G, and displacements and deformations are inversely proportional to it, $G = 1$can be assumed in the calculations.

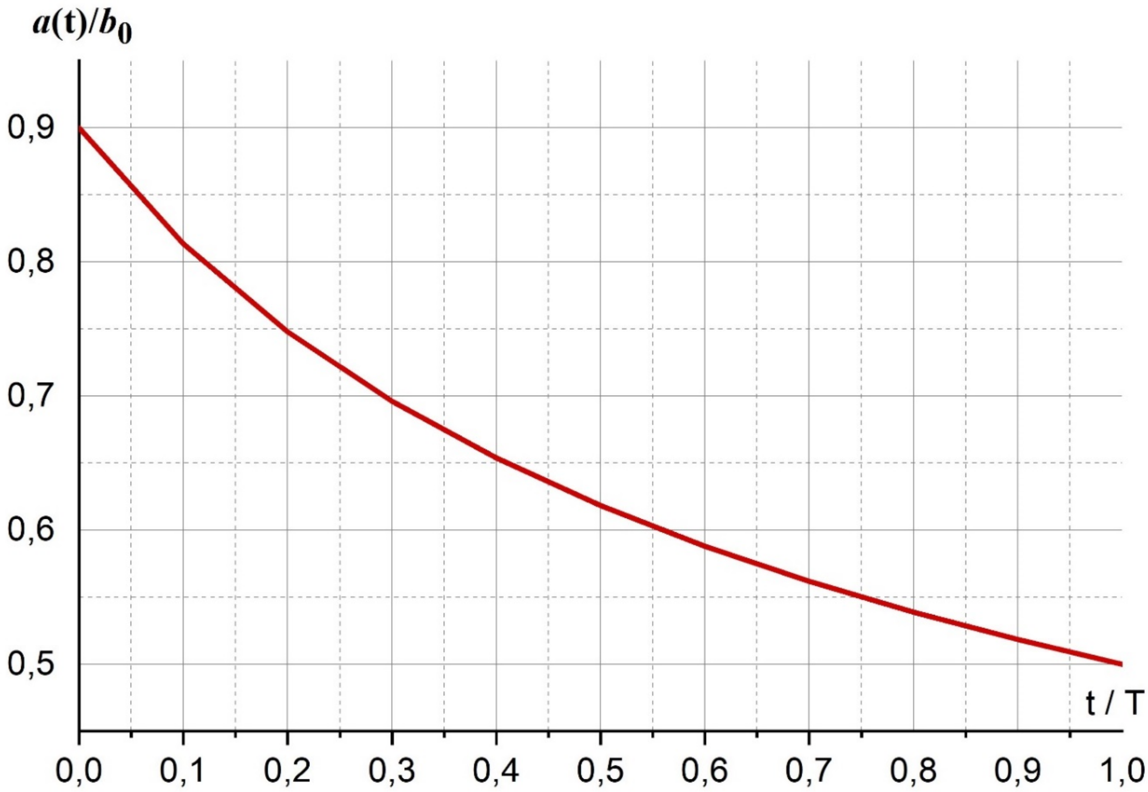


Figure 5 – The law of changing the inner radius of the cylinder

Substituting (46), (47) into relation (40) taking into account the numerical values of the parameters (material constants) included in them and the values a_0, a_1, we obtain

$$G_2(\tau) = 0{,}235 + 2{,}765\frac{\tau}{T},$$

$$\varphi_2(\tau) = 0{,}0235 + 0{,}2765\tau + \frac{207{,}412}{T}e^{0{,}018T} - \frac{207{,}412}{T}e^{(0{,}016T - 0{,}02\tau)}. \quad (48)$$

Expressions (48) are presented in dimensionless form for convenience.

Substituting (48) to (45), we determine the function $A(t)$, and by formulas (32), (33) – the stress-strained state of the cylinder.

Fig.6, 7 and 8 show the time dependences of the maximum tangential stress and displacement $u_r$at different values of the build-up time T for the following points of the cylinder: 1– $r = b_0$; 2–$r = 0{,}9b_0 - 0$; 3 – $r = 0{,}79b_0$.

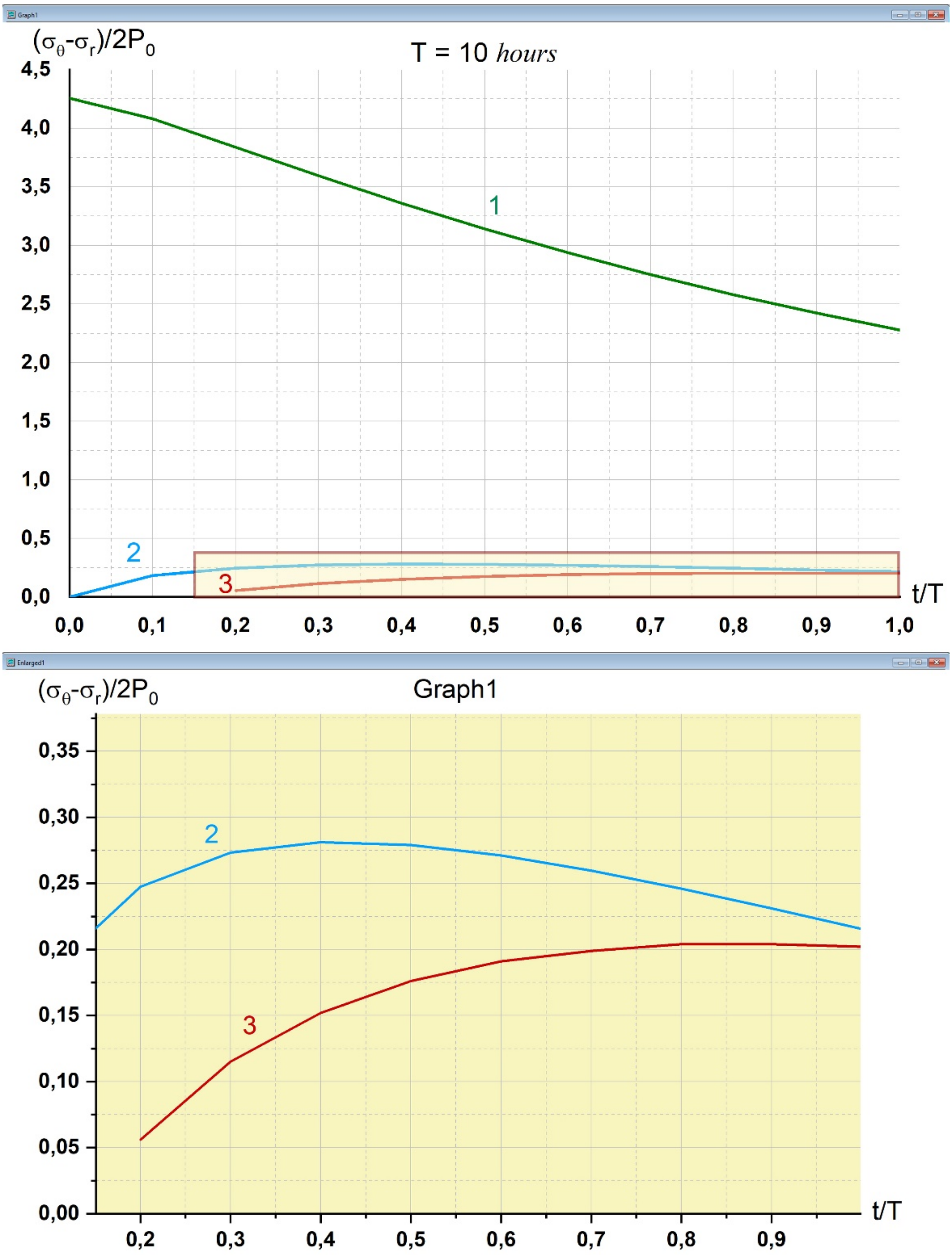
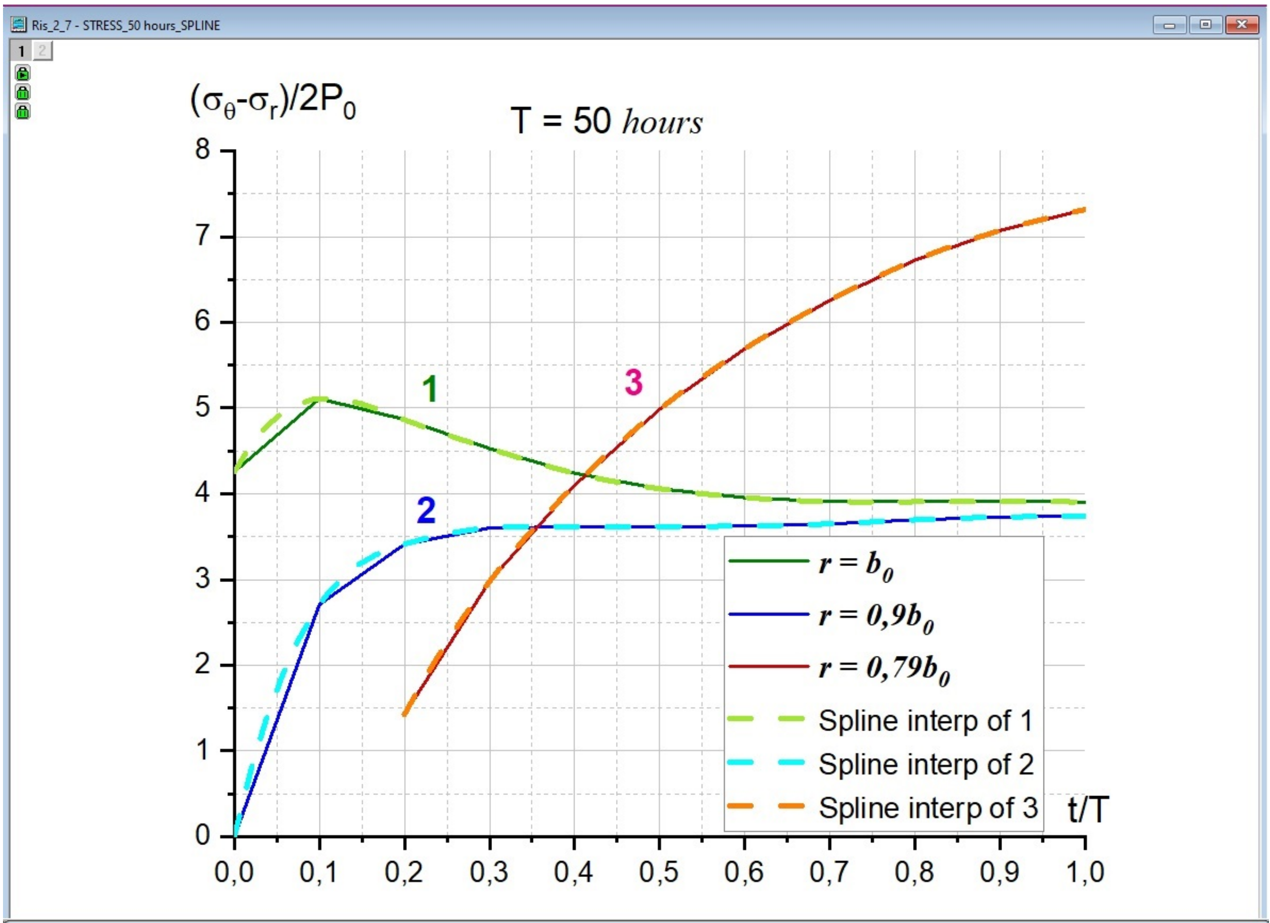


Figure 6 – Dependence of the maximum tangential stresses on time for different points of the cross-section of the cylinder at $T = 10$ hours

For a more visual presentation of the obtained results, in the lower part (Enlarged1) of the graph, Fig.6 shows an enlarged fragment highlighted in colour in the upper part (Graph1).



Ris27 - STRESS_50 hours_SPLINE

	A(X1)	B(Y1)	C(Y1)	D(Y1)	E(Y#)	F(Y1)	G(Y#)	H(Y1)	I(Y#)	J(Y1)
Long Name					Interpolate	Coefficients	Interpolate	Coefficients	Interpolate	Coefficient
Units										
Comments					Spline interp of 1	Spline interp of 1	Spline interp of 2	Spline interp of 2	Spline interp of 3	Spline interp of 3
F(x)=										
1	0	4,25532	0	--	4,25532	-232,71379	0	-380,30363	1,425	-66,63615
2	0,1	5,10983	2,70879	--	4,5525	-109,81174	0,735	-200,6453	1,75003	-43,1
3	0,2	4,86621	3,41113	1,425	4,77768	13,09031	1,351	-20,98697	2,05445	-19,56385
4	0,3	4,52802	3,60201	2,978	4,93867	0,70035	1,85939	-22,28878	2,33967	-19,04462
5	0,4	4,24014	3,6156	4,1	5,04327	14,29719	2,27159	3,7745	2,6071	-14,05769
6	0,5	4,05962	3,61653	4,988	5,09928	6,52616	2,59901	-0,40431	2,85816	-8,12462
7	0,6	3,9558	3,6241	5,693	5,11451	5,62278	2,85306	1,82281	3,09427	-13,44385
8	0,7	3,90929	3,64899	6,259	5,09676	5,36073	3,04514	3,50623	3,31684	-10,1
9	0,8	3,90693	3,696	6,725	5,05385	-0,56783	3,18667	-2,57244	3,52728	-6,75615
10	0,9	3,90878	3,72855	7,071	4,99357	-0,57137	3,28906	-1,8971	3,72701	
11	1	3,90491	3,74212	7,316	4,92372	-0,5749	3,36371	-1,22177	3,91745	
12					4,85212		3,42203		4,1	

Sheet1

Figure 7 – Dependence of the maximum tangential stresses on time for different points of the cross-section of the cylinder at $T = 50$ hours

For the convenience of using the obtained results in practice, in Fig.7, the lower part (Ris27_STRESS_50 hours_SPLINE) shows additional calculations in the OriginPro 2019b [22], package, which are related to interpolation (dashed lines in the upper part) by cubic splines.

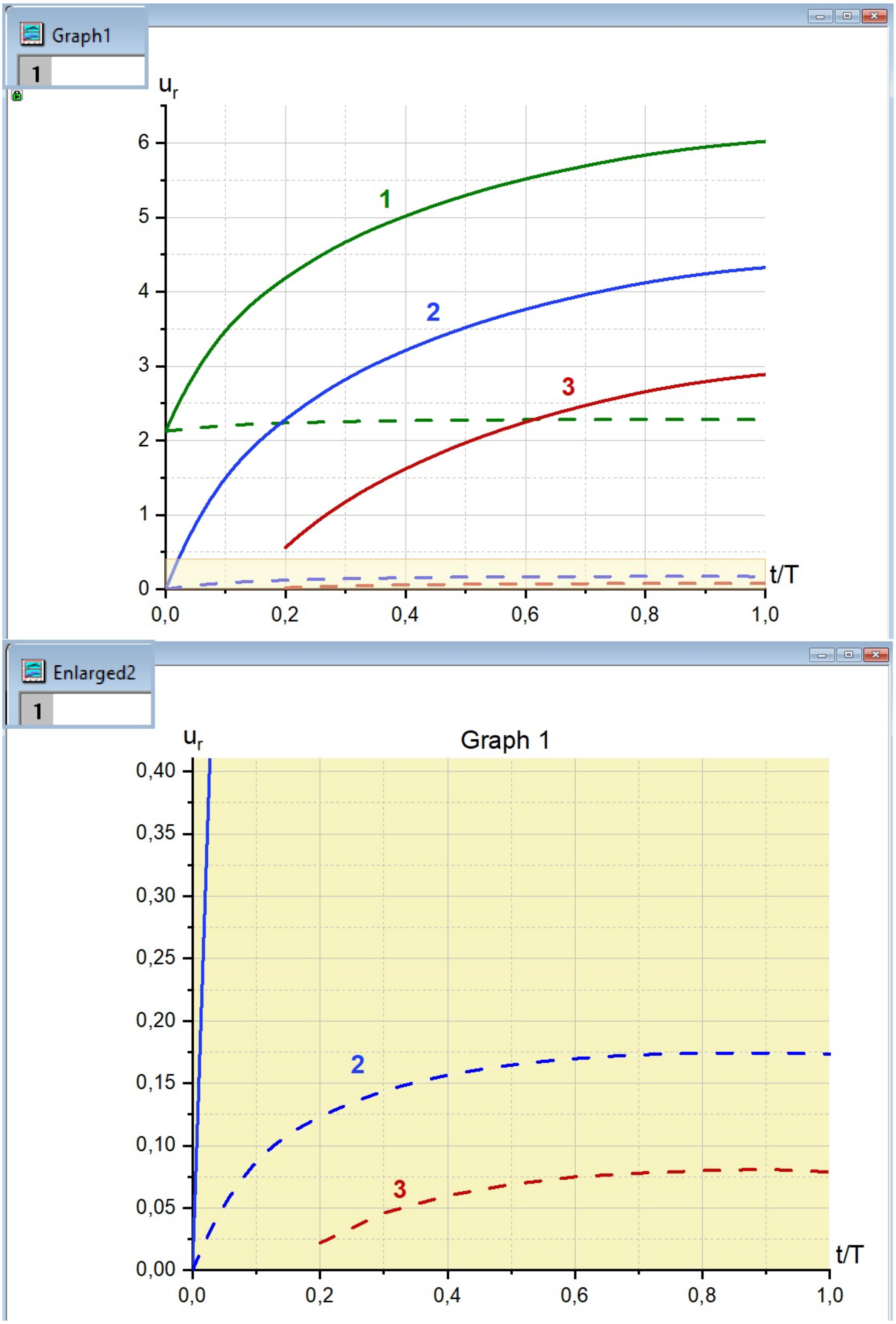


Figure 8 – Dependence of radial displacements on time for different points of the cross-section of the cylinder at $T = 50$ hours (solid lines) and $T = 10$ hours (dashed lines)

As we can see from Fig.6 – 8, the duration of the build-up process affects both the stress state and the displacement of the points of the cylinder. At the same time, the nature of the curves changes slightly, but the magnitudes of the stresses and displacements themselves change strongly. So, for example, from Fig.6, 7, it is obvious that with ten hours of build-up, the stresses at point *1* monotonically decrease, and at point *2* they increase slightly, and by the end of the build-up process, they even decrease little by little. At fifty hours of build-up, the picture is completely different – the stresses at the second point begin to increase significantly later and approach the values at point *1*, and the stresses at point *3* increase so much that they already exceed the stresses at points *1* and *2* twice. This stress distribution can be explained as follows: firstly, points *2* and *3* are always closer to the line of action of the load than point *1*, secondly, with a faster build-up process, the material does not have time to fully reveal its viscoelastic properties, which is also confirmed by the displacement curves shown in Fig.8.

Thus, the presented calculation scheme makes it possible to estimate the influence of the build-up time on the stress-strain state of the growing cylinder.

5.5. Effects of component material heterogeneity on the stress-strain state during the build-up and Analysis of the obtained results

One of the important aspects of interest, both in the mechanics of growing viscoelastic bodies and in the production of various parts by the methods of build-up, is the issue of how the stress-strain state is formed in the part when the original and build-up material have different viscoelastic properties, i.e. the part obtained in this way consists of a non-homogeneous, as a whole, material [9].

We consider an empty circular viscoelastic (elastic moment modulus of the material – $G^{(1)}$) cylinder of inner radius a and outer radius b_0 (Fig.9). Starting from the moment of time $t = 0$, an internal pressure $P(t)$, $P(0) = P_0$ is applied to the cylinder from the inside, and its continuous increase begins with a material with a different, generally speaking, elastic moment modulus $G^{(2)}$, so that the outer radius of the cylinder increases monotonically according to the law $b(t),\ b(0) = b_0$. In the future, in this paragraph, the superscripts "1" and "2", taken in parentheses, will denote the values corresponding to the initial empty cylinder ($a \leq r \leq b_0$.) and the growing region ($b_0 < r \leq b(t)$). The absence of a superscript means that the quantity under consideration is defined at $a \leq r \leq b(t)$. At the moment of time T, the outer radius of the cylinder reaches the value b_1, and the growth process stops, i.e. $b(t) = b_1$ at $t \geq T$.

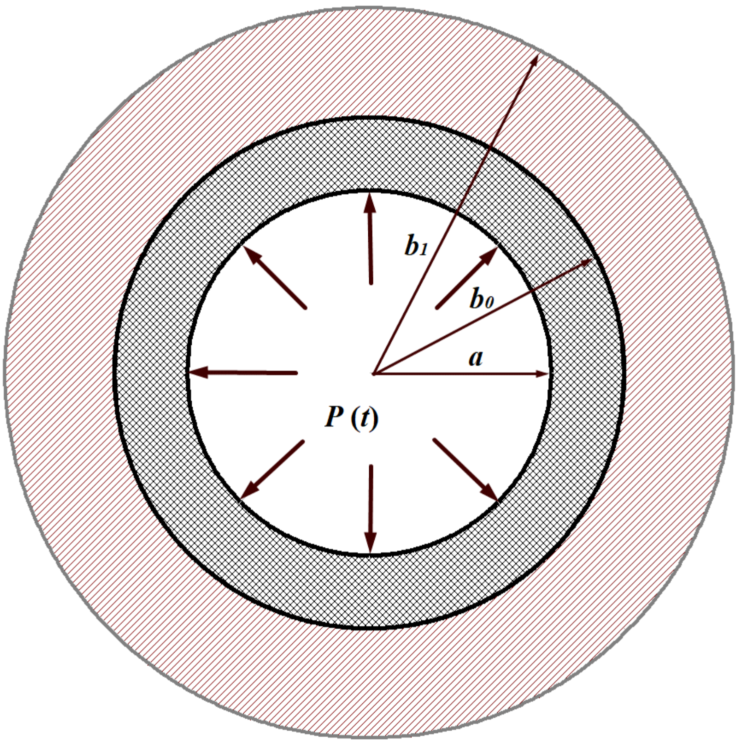


Figure 9 – Build-up of a folded cylinder

The functions $b(t)$ and $P(t)$are assumed to be continuously differentiable on the interval $0 < t < T$. It is necessary to determine the stress-strain state in the cylinder for all $t \geq 0$.

In [11], the problem of the build-up of a cylinder is considered assuming the same viscoelastic characteristics of the original and built-up material. The purpose of this paragraph is to study the effects of various viscoelastic characteristics of the body and its build-up time on the stress-strain state. The corresponding ratios are obtained using the approach outlined in Section 3 of this work, as a result of which, after comparing the obtained results with the known ones, we are able to talk about the probability of the results as a whole for the chapter.

We enter the cylindrical coordinates r, θ, z and consider the plane deformation of the cylinder, that is, we set $u_z = 0$.

We write down the main equations of the problem:

$$\varepsilon_r + \varepsilon_\theta = 0, \tag{49}$$

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$$\frac{\partial\sigma_r}{\partial r} = -\frac{\sigma_r - \sigma_\theta}{r}, \tag{50}$$

$$\dot{\varepsilon}_r = \frac{\partial \dot{u}_r}{\partial r}, \quad \dot{\varepsilon}_\theta = \frac{\dot{u}_r}{r}. \tag{51}$$

Here the dot indicates the partial derivative over time.

We write the rheological equations for the material of the initial cylinder and the build-up area in a form similar to (15) in the case of the linear creep law:

$$\sigma_r^{(1)}(t,r) - \sigma_\vartheta^{(1)}(t,r) = 2G^{(1)}(t-\tau^*)\left(\varepsilon_r^{(1)}(t,r) - \varepsilon_\vartheta^{(1)}(t,r)\right) -$$

$$-\int_{\tau^*}^{t} R^{(1)}(t-\tau^*, \tau-\tau^*)\left(\varepsilon_r^{(1)}(\tau,r) - \varepsilon_\vartheta^{(1)}(\tau,r)\right)d\tau,$$

$$\sigma_r^{(2)}(t,r) - \sigma_\vartheta^{(2)}(t,r) = 2G^{(2)}(t-\tau^*)\left(\varepsilon_r^{(2)}(t,r) - \varepsilon_\vartheta^{(2)}(t,r)\right) -$$

$$-\int_{\tau^*}^{t} R^{(2)}(t-\tau^*, \tau-\tau^*)\left(\varepsilon_r^{(2)}(\tau,r) - \varepsilon_\vartheta^{(2)}(\tau,r)\right)d\tau.$$

Here, as before, $\tau^* = \tau^*(r)$ is the nucleation time of the elementary layer of the cylinder; τ is the age of the material in which the load was applied to it; $G^{(1)}$, $G^{(2)}$ are elastic moment moduli of the material; $R^{(1)}(t,\tau)$, $R^{(2)}(t,\tau)$ are the corresponding relaxation cores.

The function $\tau^* = \tau^*(r)$ is zero at $a \le r \le b_0$and coincides with the inverse function of the function $b(t)$, at $b_0 \le r \le b_1$.

The boundary conditions have the form

$$\sigma_r^{(1)}(t,a) = -P(t), \ \ \sigma_r^{(1)}(0,b_0) = 0, \ \ \sigma_r^{(2)}\big(t,b(t)\big) = 0 \ \ (0 < t \le T),$$
$$\sigma_r^{(2)}(t,b_1) = 0 \ \ (t > T), \ \ \sigma_\vartheta^{(2)}\big(t,b(t)\big) = 0 \ \ (0 < t \le T).$$

After transformations similar to items 3 and 4, the stress-strain state in the considered cylinder is determined by the formulas [9]:

$$u_r^{(1)}(t,r) = \frac{A(t)}{r}, \qquad -\varepsilon_r^{(1)}(t,r) = \varepsilon_\vartheta^{(1)}(t,r) = \frac{A(t)}{r^2},$$

$$\sigma_r^{(1)}(t,r) - \sigma_\vartheta^{(1)}(t,r) = -\frac{4G^{(1)}(t)}{r^2}\left(A(t) - \int_0^t R^{(1)}(t,\tau)A(\tau)d\tau\right),$$

$$\sigma_r^{(1)}(t,r) =$$

$$= -P(t) + 2G^{(1)}\left(\frac{1}{a^2} - \frac{1}{r^2}\right)\left[A(t) - \int_0^t R^{(1)}(t,\tau)A(\tau)d\tau\right], a \le r \le b_0; \tag{52}$$

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$$u_r^{(2)}(t,r)=\frac{A(t)-A(\tau^*(r))}{r},\qquad -\varepsilon_r^{(2)}(t,r)=\varepsilon_\vartheta^{(2)}(t,r)=\frac{\left(A(t)-A(\tau^*(r))\right)}{r^2},$$

$$\sigma_r^{(2)}(t,r)-\sigma_\vartheta^{(2)}(t,r)=$$

$$=-\frac{G^{(2)}(t-\tau^*)}{G^{(2)}(0)}\sigma_\vartheta^{(2)}(\tau^*(r),r)\left[1-\int_{\tau^*}^{t}R^{(2)}(t-\tau^*,\tau-\tau^*)d\tau\right]-$$

$$-\frac{4}{r^2}G^{(2)}(t-\tau^*)\left\{A(t)-A(\tau^*)-\int_{\tau^*}^{t}R^{(2)}(t-\tau^*,\tau-\tau^*)\left(A(\tau)-A(\tau^*)\right)d\tau\right]\Bigg\},$$

$$\sigma_r^{(2)}(t,r)=-\frac{2}{G^{(2)}(0)}\int_r^{b(t)}\sigma_\vartheta^{(2)}(\tau^*(\rho),\rho)G^{(2)}(t-\tau^*)\left[1-\int_{\tau^*}^{t}R^{(2)}(t-\tau^*,\tau-\tau^*)d\tau\right]\frac{d\rho}{\rho}-$$

$$-8\int_r^{b(t)}G^{(2)}(t-\tau^*)\left\{A(t)-A(\tau^*)-\int_{\tau*}^{t}R^{(2)}(t-\tau^*,\tau-\tau^*)\left(A(\tau)-A(\tau^*)\right)d\tau\right\}\frac{d\rho}{\rho^3},$$

$$b_0\le r\le b(t).\qquad (53)$$

In formulas (52), (53), the function $A(t)$ is determined from the solution of the integral equation obtained from the condition of continuity of the radial stress component at the interface between the output cylinder and the build-up area.

For the convenience of numerical calculations, we assume that $G^{(1)}=const$, $G^{(2)}=const$, and refer to all values with the dimension of length to the outer radius $b_0$of the output cylinder, and the values with the dimension of stress – to the shear modulus $G^{(1)}$ of the output cylinder. We assume that time is measured in dimensionless conventional units. Then, keeping the previous notations for dimensionless quantities, the integral equation for determining the function $A(t)$ has the form

$$D_1(t)\,A(t)-\int_0^t A(\tau)\,D_2(t,\tau)\,d\tau=Y(t),\qquad (54)$$

where notations are entered

$$D_1(t)=\frac{1}{2}\left(\frac{1}{a^2}-1\right)+\frac{G^*}{2}\left(1-\frac{1}{b^2(t)}\right),$$

$$D_2(t,\tau)=\frac{1}{2}\left(\frac{1}{a^2}-1\right)R^{(1)}(t,\tau)+G^*\frac{\dot b(\tau)}{b^3(\tau)}\left[1-\int_\tau^t R^{(2)}(t-\tau,s-\tau)ds\right]+$$

$$+G^* \int_0^{\tau} R^{(2)}(t-s,\tau-s)\frac{\dot{b}(s)}{b^3(s)}ds,$$

$$Y(t)=\frac{1}{8}p(t)-\frac{1}{4}\int_1^{b(t)} \sigma_\vartheta^*(\tau^*(\rho),\rho)\left[1-\int_{\tau^*}^{t} R^{(1)}(t-\tau^*,\tau-\tau^*)\,d\tau\right]\frac{d\rho}{\rho}.$$

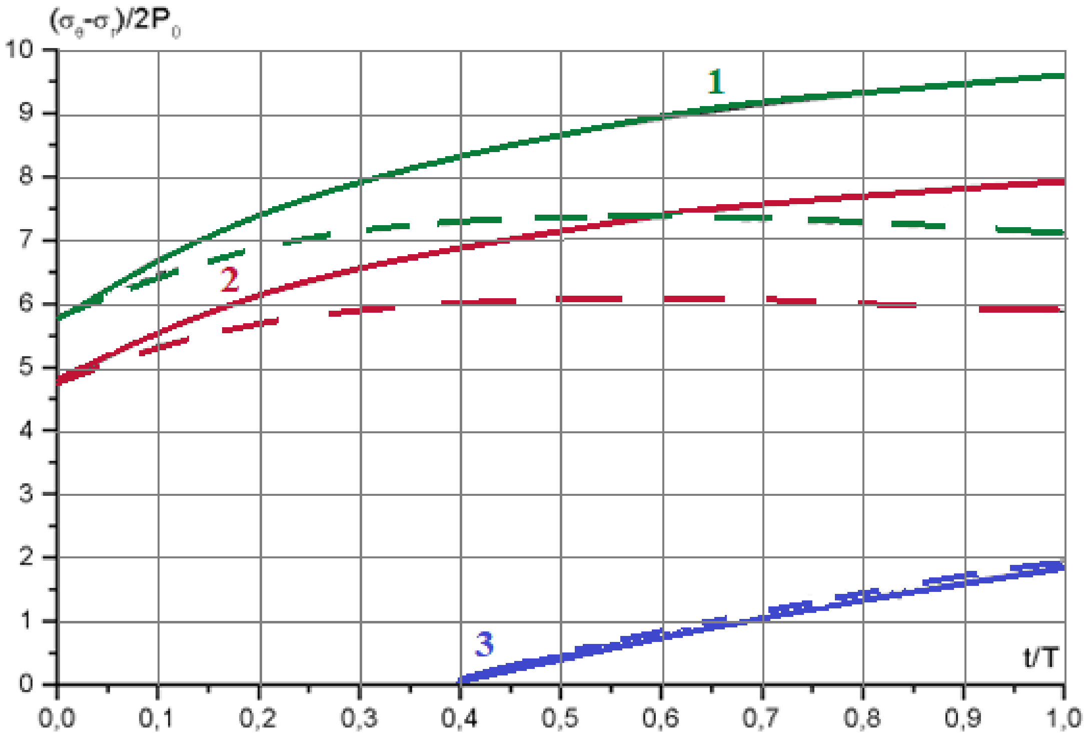

Figure 10. Time dependence of the maximum tangential stresses in a growing compound viscoelastic cylinder at different points of the cross-section at $T = 10$ days

As an example, we consider the problem related to the manufacture of polymer cylinders for work in aggressive environments, with an outer layer of viscoelastic material, which are made by winding (building up) the material on the initial cylindrical shells of elastic material. It is quite natural that in this case the viscoelastic properties of the output cylinder can be neglected and $R^{(1)}(t,\tau)\equiv 0$can be accepted. Other initial data were chosen as in clause 4. Equation (54) was solved by the method of successive approximations. The obtained results are shown in Fig.10 – 13. They show the dependences of the maximum tangential stress and displacement $u_r$on time for the following points of the cylinder: curve *1* corresponds to the point $r=a$; curve *2* – $r=b_0=1.1a$; curve *3* – $r=1.21a$. During calculations, parameters C_0, A_0, β, γ

ISBN 978-3-949059-59-9

and values b_0, b_1 are fixed and equal: $C_0 = 0.05$, $A_0 = 0.75$, $\beta = 0.02 days^{-1}$, $\gamma = 0.1 days^{-1}$, $b_0 = 1.1a$, $b_1 = 1.5a$ the pressure changes according to the law $P(t) = P_0 + \frac{P_0 t}{T}$, the outer radius increases according to the dependence

$$\frac{1}{b^2(t)} = \frac{1}{b_0^2} + \left(\frac{1}{b_1^2} - \frac{1}{b_0^2}\right)\frac{t}{T}, \qquad 0 \le t \le T.$$

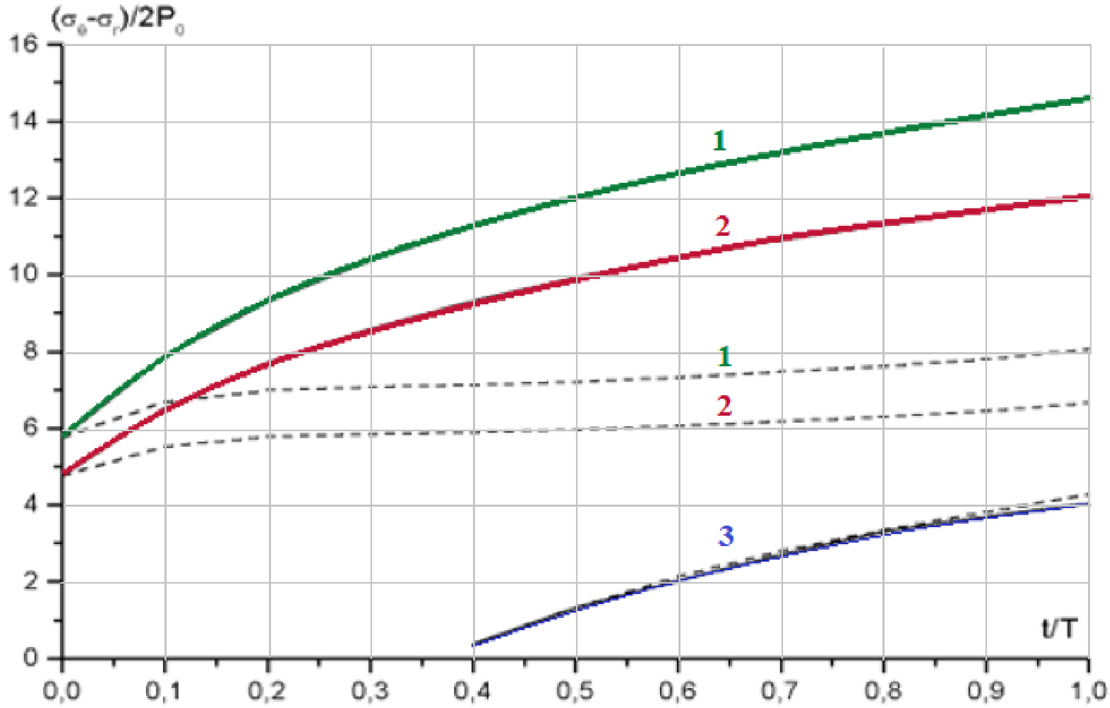

Figure 11. Dependence of maximum tangential stresses on time in a growing compound viscoelastic cylinder at different points of the cross-section at $T = 50$ days

Dashed curves are obtained for a cylinder made of a homogeneous material [11], continuous curves characterize the behaviour of a composite cylinder, the initial part of which has an elastic moment modulus that exceeds the modulus of the building material. The calculations were performed for the relative values of the modulus G^*, which was equal to the ratio of the spring-moment moduli of the original and build-up materials, and equal to 1, 2, 3. As can be seen from Fig.10 – 13, heterogeneity affects both the stress state and the displacement of points of the cylinder. At the same time, for point *3* – $r = 1.21a$, the influence of heterogeneity on stress is minimal. Fig.10, 11 show that for a non-homogeneous cylinder, an increase in the build-up time leads to a more significant increase in stresses than for a homogeneous one.

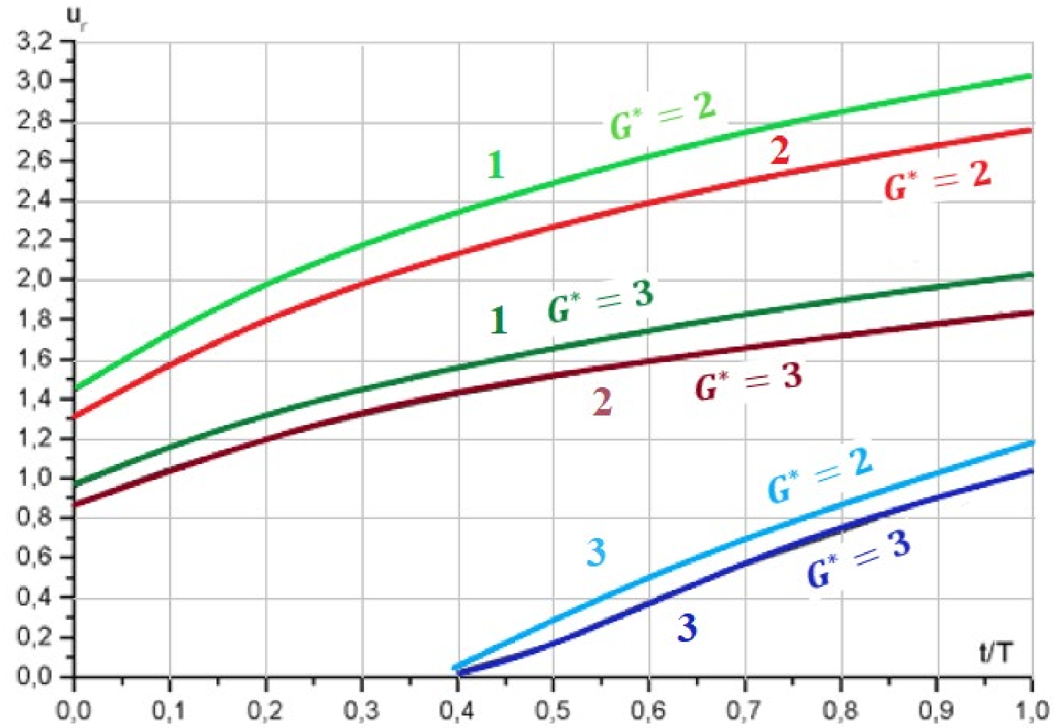

Figure 12. Distribution of radial displacements over time for different values of the ratio of the elastic moment modulus of the initial and build-up material at $T = 10$ days

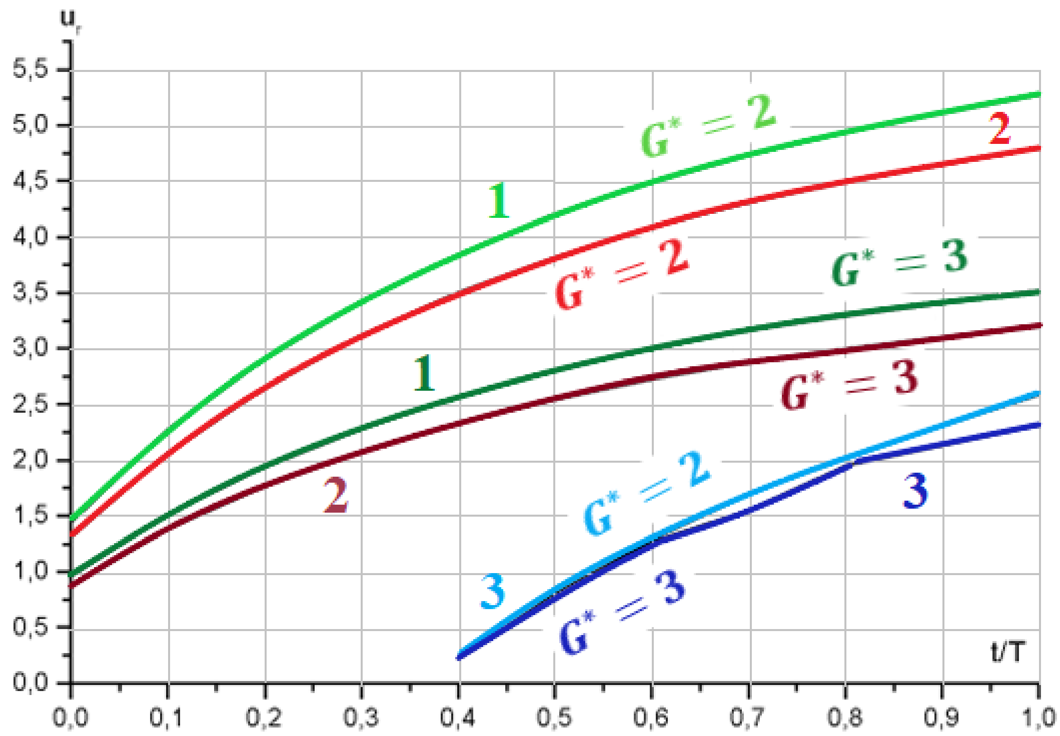

Figure 13. Distribution of radial displacements over time for different values of the ratio of the elastic moment modulus of the initial and build-up material at $T = 50$ days

Thus, the maximum stress at the point $r = a$ for a homogeneous cylinder increases by *1.1* times when the build-up duration increases from ten to fifty days, and by *1.52* times in a non-homogeneous cylinder $(G^* = 2)$. The graphs shown in Fig.12 and 13 demonstrate that increasing the stiffness of the material of the initial cylinder leads to a decrease in the movement of all its points during the build-up process. At the same time, an increase in the build-up time leads to a general increase in displacements but does not change the degree of influence of the stiffness of the initial material of the cylinder on the final displacement values. So, for example, with ten days of build-up, displacements at $G^* = 3$decreased compared to displacements at $G^* = 2$ by *33%*, and the same value at fifty days of build-up is *34%*.

In addition, regression analysis was performed for tangential stresses during 10 hours of build-up, namely, nonlinear cubic pairwise regression. Calculations were made in Microsoft Excel. The results of the calculations are shown in Fig.14.

The following conclusions can be drawn based on the obtained results of mathematical interpretation.

1. Tangential stresses in layers 1 and 2 (models 1(2) and 2(1)) affect each other differently. So, for example, in the case of model 1(2), the influence of stresses in layer 2 on stresses in layer 1 is moderate and the regression equation is difficult to recommend 100% for practical problems. Only under certain conditions. In the opposite case, model 2(1), the relationship is very close, and the cubic regression equation can be considered the optimal model and used in design institutes, design bureaus and other institutions.

2. The connection between the tangential stresses in layers 1 and 3 (models 1(3) and 3(1)) is close in both cases. Similarly, the cubic regression equation in both models is estimated with high accuracy. They can be used in the simulation of manufacturing processes of cylindrical parts and structural elements by means of build-up methods, predicting stress-deformed state.

3. The connection between the tangential stresses in layers 2 and 3 (models 2(3) and 3(2)) turned out to be unpredictable. Analysis of the results shows that the stresses in layer 3 significantly influence the stresses in layer 2 (model 2(3)), the connection between them is close, and the cubic regression equation is highly accurate. It can be used in mathematical modeling. In the case of model 3(2), there is no connection between stresses at all. With the help of nonlinear paired regression, we can get any approximation.

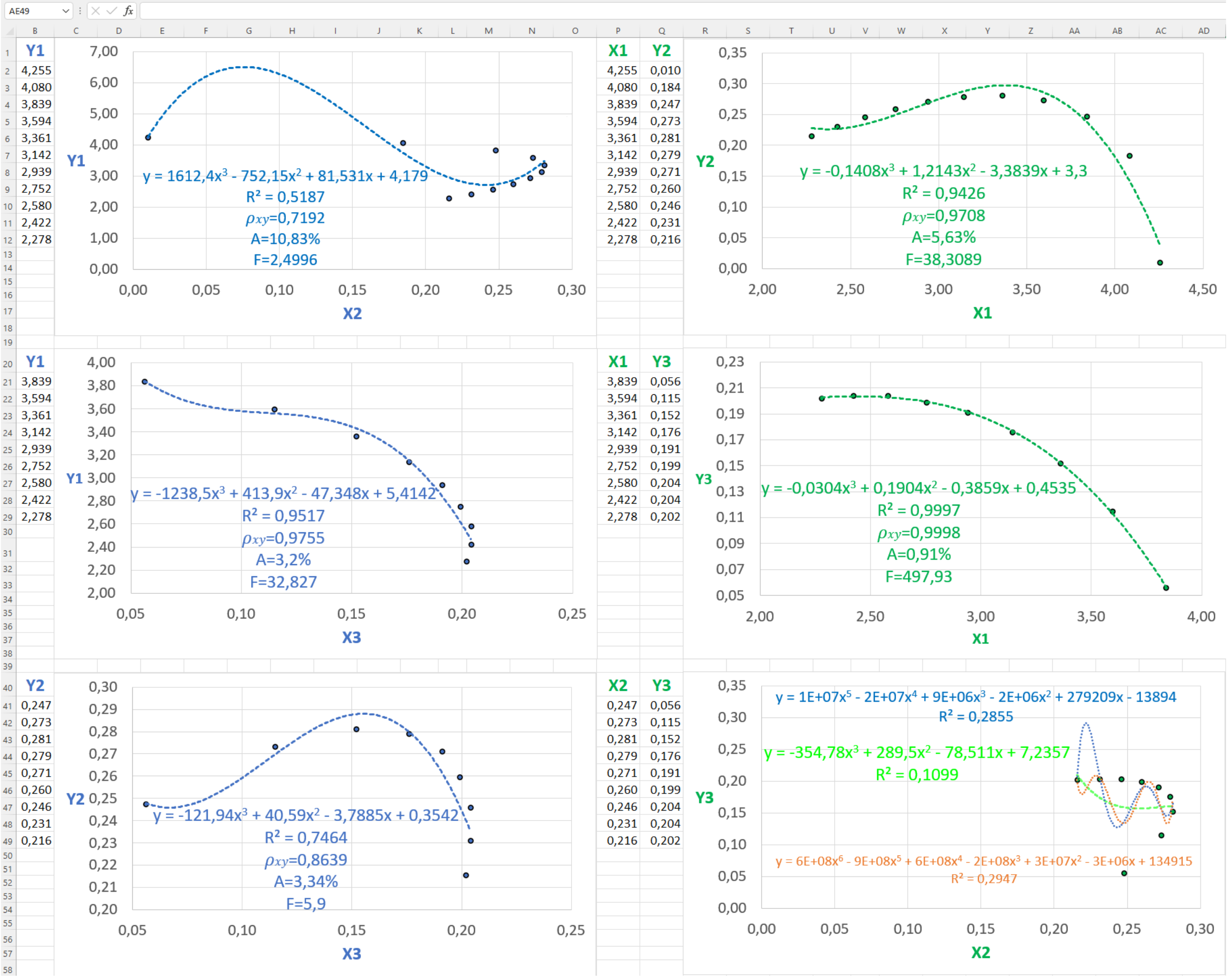

Figure 14 – Regression analysis

4. Having experimental data in one of the layers of the cylinder, it is possible to predict the behavior of stresses in other layers of the part without conducting additional experiments, which saves money and time.

It is shown how with the help of nonlinear paired regression, namely cubic regression, having experimental data, it is possible to investigate the connection of tangential stresses between different layers of a cylindrical part during build-up. Despite some contradictions that arose during the evaluation of the models according to various regression parameters, it was established that at least four of the six models are optimal and allow to adequately model the process of the formation of the stress-strain state in the details and elements of structures with significantly lower costs even before the stage of manufacturing finished products.

Conclusions

The conducted research on the formation of the stress-strain state in growing bodies, as well as on the influence of various parameters of the technological process on it, showed that the proposed model adequately describes the real process of manufacturing parts by the methods of build-up. The developed algorithm allows solving the problems of both external and internal build-up taking into account the degree of nonlinearity of the creep law. The algorithm has a uniform calculation scheme and uniformity in the direction of increase. In addition, the unified algorithm makes it possible to solve the problems of determining the stress-strain state in parts when the original and build-up material have different viscoelastic properties, that is, the part obtained in this way consists of generally heterogeneous material.

The obtained results make it possible to evaluate the influence of both the build-up time and viscoelastic characteristics on the stress-strain state of bodies of rotation made of viscoelastic material.

The comparison of the obtained results of the research of the effect of various viscoelastic characteristics of the body and its build-up time on the stress-strain state with known solutions to the problem of building up a cylinder with the same viscoelastic characteristics of the original and build-up material [11] shows a satisfactory agreement.

This agreement and the gained analytical ratios confirm the reliability of the results as a whole for the chapter.