KAPITEL 11 / CHAPTER 11 [11]
MATHEMATICAL MODELING OF CRITICAL PHENOMENA ACCORDING TO THE PLEBIANSKY-DEMYANSKY METRIC

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Introduction

Modern ideas about the evolution of the Universe are increasingly associated with phase transitions of space-time from one stable phase to another stable phase. Such transitions can occur due to resonance phenomena within the existing stable phase. Under certain conditions associated with changes in the state of space-time, a stable phase can become unstable, which, in turn, can lead to a phase transition to a more stable state. Of particular interest are states corresponding to the same levels of potential functions describing the system under study. Such phase states correspond to the conditions for the simultaneous coexistence of two phases. The slightest changes within one of the phases can lead to a loss of system stability and a transition to one of the coexisting phases. Mathematical modeling of the processes of phase transitions, which are described by modern cosmology and the theory of gravity, is associated with great difficulties. The fact is that when describing the states of space-time, complex systems of differential equations arise, the analytical solution of which is associated with enormous difficulties. This problem can be partially solved by using modern computer algebra systems, which make it possible to perform labor-intensive analytical calculations.

Plebański and Demiański [1] within the framework of the Newman-Penrose method discovered a family of solutions to the Einstein-Maxwell equations, which are generalizations of the Schwarzschild problem, but the analysis of the phase states of systems that correspond to the Plebański-Demiański metric has not been carried out to date. Modern mathematical modeling methods and software make it possible to simulate the processes of emergence under various conditions of multicomponent phases of the studied systems. Recently, significant progress has been made in understanding the problem of self-organized and orderly structures of matter. To analyze the processes of formation of self-organizingly arising ordered structures, multidimensional phase diagrams are calculated, which take into account the possibility of the existence of bifurcation spaces, critical spaces, and spaces of coexistence of phases of different orders.

In order to study the phase states and critical phenomena of the island system, the

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well-known Plebański-Demiański metric in real $x^i = (p, q, \sigma, \tau)$ coordinates was presented in the form:

$$ds^2 = \frac{1}{(p+q)^2}\{\frac{1+(pq)^2}{P}dp^2 + \frac{P}{1+(pq)^2}(d\sigma + q^2 d\tau)^2 + \frac{1+(pq)^2}{Q}dq^2 - \frac{Q}{1+(pq)^2}(d\tau - p^2 d\sigma)^2\}, \quad (1)$$

where the real functions of the coordinates are $P = P(p),\ \ Q = Q(q)$ and the signature has the form (+, +, +, –) based on the gradient vectors:

$$m_i = \frac{\partial x^0}{\partial x^i} = (1,0,0,0),\ n_i = \frac{\partial x^1}{\partial x^i} = (0,1,0,0),\ p_i = \frac{\partial x^2}{\partial x^i} = (0,0,1,0),\ s_i = \frac{\partial x^3}{\partial x^i} = (0,0,0,1) \quad (2)$$

According to the solution of Plebański and Demiański [1 – 3], the energy-momentum tensor was written in the form:

$$T^{ij} = \begin{pmatrix} -(\alpha+\beta)\frac{(p+q)^4}{(1+(pq)^2)^2} & 0 & 0 & 0 \\ 0 & -(\alpha+\beta)\frac{(p+q)^4}{(1+(pq)^2)^2} & 0 & 0 \\ 0 & 0 & (\alpha+\beta)\frac{(p+q)^4}{(1+(pq)^2)^2} & 0 \\ 0 & 0 & 0 & (\alpha+\beta)\frac{(p+q)^4}{(1+(pq)^2)^2} \end{pmatrix}, (3)$$

where $\alpha,\ \beta$ are constants, and

$$P(p) = (-(\lambda/6) - \alpha) + 2np - \varepsilon p^2 + 2mp^3 + (-(\lambda/6) - \beta)p^4$$
$$Q(q) = (-(\lambda/6) - \alpha) + 2nq - \varepsilon q^2 + 2mq^3 + (-(\lambda/6) - \beta)q^4, \quad (4)$$

and $\lambda, n, \varepsilon, m$ are constants.

The purpose of the work.

The aim of the work is research based on the theory of bifurcations and the theory of catastrophes of the phase states of island systems (1) - (3), which correspond to the Plebański-Demiański metric. The goal was to find out the existence of spaces of the phase diagram in which the conditions for the emergence of bifurcation spaces, critical spaces and spaces of coexistence of phases of different orders are fulfilled. The components of the energy-momentum tensor obtained in the solutions of Plebański and Demiański are considered as the object of research. Based on the differential-topological approach, a model of the process of formation of critical spaces and the space of phase coexistence by electrovacuum island systems with local curvature flows was considered. It was believed that the phase coexistence space occurs when one stable state coexists with another stable state. The appearance of such a space is a phase transition of the first kind, determined by Maxwell's principle, when two or more global minima of the potential function have the same depth. A bifurcation space was considered a set of points where a stable phase can become unstable, and the emergence of a critical space of order two was considered a consequence of the existence of two simultaneously coexisting phases in some space.

Analysis of the conditions for the emergence of a stable phase.

To determine the spaces in which the stability conditions are met on the phase diagram of the T^{ij} components, a system of equations and inequalities was solved [4-12]:

$$\frac{\partial T^{ij}}{\partial p}=0;\ \frac{\partial T^{ij}}{\partial q}=0;\quad \det\frac{d^2T^{ij}}{dX^2}>0\,, \tag{5}$$

where the expressions $\frac{\partial T^{ij}}{\partial p}$, $\frac{\partial T^{ij}}{\partial q}$ denote the partial derivatives of the T^{ij} of the energy-momentum tensor with respect to independent p and q coordinates. To calculate the zero contours of the first partial derivative components x and y, analytical expressions were found, which were presented in the form:

$$\frac{\partial T^{11}}{\partial p}=\frac{\partial T^{22}}{\partial p}=4\frac{CB^3}{A^2}(1-\frac{q^2Bp}{A}),\qquad \frac{\partial T^{11}}{\partial q}=\frac{\partial T^{22}}{\partial q}=4\frac{CB^3}{A^2}(1-\frac{p^2Bq}{A})\,, \tag{6}$$

where $C=(-\beta-\alpha),\quad A=p^2q^2+1,\quad B=q+p$.

In system (5), the expression $\det\frac{d^2T}{dX^2}$ denotes the determinant of the matrix of the second partial derivative components of the energy-momentum tensor T^{ij} along the coordinates p and q , that is, along $X(p,q)$. To solve system (5), its analytical form was found and presented in the form:

$$\det\frac{d^2T^{11}}{dX^2}=\det\frac{d^2T^{22}}{dX^2}=\frac{16C^2B^4}{A^4}[3-\frac{p^2}{A}B(B+8q)+\frac{6p^4q^2B^2}{A^2}][3-\frac{q^2}{A}B(B+8p)+\frac{6p^2q^4B^2}{A^2}]$$
$$-\frac{144C^2B^4}{A^4}[1-\frac{2pq}{A}(1-\frac{p^2q^2B^2}{A})]^2 \tag{7}$$
$$\det\frac{d^2T^{33}}{dX^2}=\det\frac{d^2T^{44}}{dX^2}=-\det\frac{d^2T^{11}}{dX^2}$$

The matrix of partial derivatives of the momentum energy tensor used in (5) for the analysis of the space signature of the phase diagram of the existence of the island system corresponding to (1) – (3) was constructed according to the rule:

$$\frac{d^2T^{ij}}{dX^2} = \begin{pmatrix} \frac{\partial^2 T^{ij}(p,q)}{\partial p^2} & \frac{\partial^2 T^{ij}(p,q)}{\partial p \partial q} \\ \frac{\partial^2 T^{ij}(p,q)}{\partial q \partial p} & \frac{\partial^2 T^{ij}(p,q)}{\partial q^2} \end{pmatrix} \quad (8)$$

According to (8), to solve the system of equations and inequalities (5), the analysis of the Hessian matrix was performed, that is, the zero contours of the Hessian values, its topology and signature were found. The found positions of zero contours were plotted on the phase diagram of the existence of the studied island system (1) - (3) in $X(p,q)$ coordinates. On the obtained diagram, the spaces of positive and negative signature were determined. After that, the zero contours of the matrices of the first partial derivatives of the components of the energy-momentum tensor were plotted on the obtained diagram.

Based on the above, a differential-topological approach [9, 10] was used to solve system (5), according to which zero contours of derivatives (7) and their determinants were found and plotted on the phase diagram $X(p,q)$ of the state of the system under study (1) – (3). Taking into account that α, β are constants, the spaces in which the stable phase condition (5) is fulfilled were determined, as shown in Fig. 1. The zero contours of the determinants (7) and their topology were determined using the open computer algebra system MAXIMA [13] and direct calculations around the found zero contours of the energy-momentum tensor components.

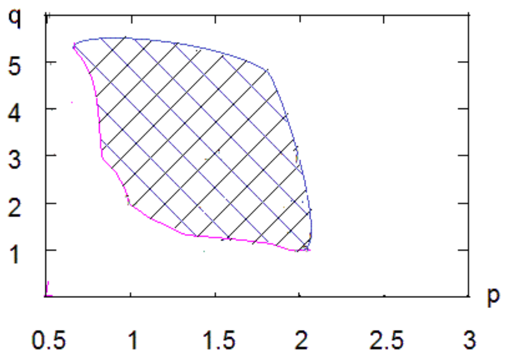

Fig. 1 - The phase diagram space in which the stable phase condition is satisfied for the T^{11} and T^{22} components.

Analysis of the conditions for the emergence of bifurcation space.

To determine the spaces in which the conditions for the existence of the bifurcation space are met on the phase diagram of the T^{ij} components, a system of equations was solved by a method similar to the solution of system (5) [4 – 12]:

$$\frac{\partial T^{ij}}{\partial p} = 0; \ \frac{\partial T^{ij}}{\partial q} = 0; \quad \det \frac{d^2 T^{ij}}{dX^2} = 0 \ . \quad (9)$$

The results of solving system (9) are shown in Fig. 2:

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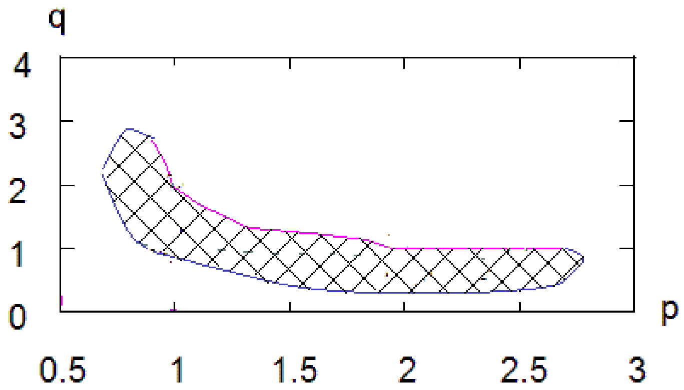

Fig. 2 - The phase diagram space in which the bifurcation space condition for T^{11} and T^{22} is satisfied.

Analysis of the conditions for the emergence of spaces of phase coexistence.

To clarify the fulfillment of the conditions for the occurrence of phase coexistence spaces, a system of equations and inequalities was solved, to which, in addition to the fulfillment of conditions (9), the conditions of zero values of the third partial derivative components of the tensor T^{ij} along the $X(p,q)$ coordinates were added, as well as the condition of a positive signature was added of the determinant of the fourth partial derivatives matrix:

$$\frac{\partial T^{ij}}{\partial p}=0;\ \frac{\partial T^{ij}}{\partial q}=0;\quad \det\frac{d^2T^{ij}}{dX^2}=0,$$

$$\frac{\partial^3T^{ij}}{\partial p^3}=0,\frac{\partial^3T^{ij}}{\partial p^2\partial q}=0,\frac{\partial^3T^{ij}}{\partial p\partial q\partial p}=0,\frac{\partial^3T^{ij}}{\partial p\partial q^2}=0,\frac{\partial^3T^{ij}}{\partial q\partial p^2}=0,\frac{\partial^3T^{ij}}{\partial q\partial p\partial q}=0,\frac{\partial^3T^{ij}}{\partial q^2\partial p}=0,\frac{\partial^3T^{ij}}{\partial q^3}=0,\quad (10)$$

$$\det\frac{d^4T^{ij}}{dX^4}>0,$$

where $\det\frac{d^4T^{ij}}{dX^4}$ is the determinant of the matrix of the fourth partial derivatives of the components of T^{ij} along the $X(p,q)$ coordinates [14]:

$$\frac{d^4T^{ij}}{dX^4}=\begin{pmatrix}\begin{pmatrix}\frac{\partial^4T^{ij}(p,q)}{\partial p^4} & \frac{\partial^4T^{ij}(p,q)}{\partial p^3\partial q}\\ \frac{\partial^4T^{ij}(p,q)}{\partial p^2\partial q\partial p} & \frac{\partial^4T^{ij}(p,q)}{\partial p^2\partial q^2}\end{pmatrix} & \begin{pmatrix}\frac{\partial^4T^{ij}(p,q)}{\partial p\partial q\partial p^2} & \frac{\partial^4T^{ij}(p,q)}{\partial p\partial q\partial p\partial q}\\ \frac{\partial^4T^{ij}(p,q)}{\partial p\partial q^2\partial p} & \frac{\partial^4T^{ij}(p,q)}{\partial p\partial q^3}\end{pmatrix}\\ \begin{pmatrix}\frac{\partial^4T^{ij}(p,q)}{\partial q\partial p^3} & \frac{\partial^4T^{ij}(p,q)}{\partial q\partial p^2\partial q}\\ \frac{\partial^4T^{ij}(p,q)}{\partial q\partial p\partial q\partial p} & \frac{\partial^4T^{ij}(p,q)}{\partial q\partial p\partial q^2}\end{pmatrix} & \begin{pmatrix}\frac{\partial^4T^{ij}(p,q)}{\partial q^2\partial p^2} & \frac{\partial^4T^{ij}(p,q)}{\partial q^2\partial p\partial q}\\ \frac{\partial^4T^{ij}(p,q)}{\partial q^3\partial p} & \frac{\partial^4T^{ij}(p,q)}{\partial q^4}\end{pmatrix}\end{pmatrix}\quad (11)$$

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System (10) defines the conditions when one stable state is replaced by another stable state. This is a first-order transition where the two minima have the same depth. The location of the points of the phase diagram where conditions (10) are fulfilled corresponds to the space of simultaneous coexistence of two phases [5, 15].

To solve the system (10), the topology of the found bifurcation space according to (9) shown in Fig. 2 was used and, first, analytical expressions of the third partial derivatives of T^{ij} were found:

$$\frac{\partial\ T^{11}}{\partial p^3} = \frac{\partial^3 T^{22}}{\partial p^3} = \frac{24CB}{A^2}(1 - \frac{2q^2B(q+4p)}{A} + \frac{3q^4B^2p(q+5p)}{A^2} - \frac{8q^6p^3B^3}{A^3}),$$

$$\frac{\partial^3 T^{11}}{\partial p^2\partial q} = \frac{\partial^3 T^{22}}{\partial p^2\partial q} = \frac{8CB}{A^2}\{3 - \frac{Bq[2B(2q+9p)+6p(2q+p)]}{A} + \frac{3B^2q^3p^2(9q+13p)}{A^2} - \frac{24p^4q^5B^3}{A^3}\},$$

$$\frac{\partial^3 T^{11}}{\partial p\partial q\partial p} = \frac{\partial^3 T^{22}}{\partial p\partial q\partial p} = \frac{8CB}{A^2}\{3 - \frac{Bq[B^2+6p(p+2q)+2B(q+4p)]}{A} + \frac{3q^3B^2p^2(9B+4p)}{A^2} - \frac{24p^4q^5B^3}{A^3}\},$$

$$\frac{\partial^3 T^{11}}{\partial p\partial q^2} = \frac{\partial^3 T^{22}}{\partial p\partial q^2} = \frac{8CB}{A^2}\{3 - \frac{Bp[B^2+6q(q+2p)+2B(4q+p)]}{A} + \frac{3p^3B^2q^2(9B+4q)}{A^2} - \frac{24p^5q^4B^3}{A^3}\},$$

$$\frac{\partial^3 T^{11}}{\partial q\partial p^2} = \frac{\partial^3 T^{22}}{\partial q\partial p^2} = \frac{8CB}{A^2}\{3 - \frac{Bq[B^2+6(2q^2+p^2)+2B(q+4p)]}{A} + \frac{3q^3B^2p^2(9B+4p)}{A^2} - \frac{24p^4q^5B^3}{A^3}\},$$

$$\frac{\partial^3 T^{11}}{\partial q\partial p\partial q} = \frac{\partial^3 T^{22}}{\partial q\partial p\partial q} = \frac{8CB}{A^2}\{3 - \frac{Bp[B^2+6q(q+2p)+2B(4q+p)]}{A} + \frac{3p^3B^2q^2(9B+4q)}{A^2} - \frac{24p^5q^4B^3}{A^3}\}, \quad (12)$$

$$\frac{\partial^3 T^{11}}{\partial q^2\partial p} = \frac{\partial^3 T^{22}}{\partial q^2\partial p} = \frac{8CB}{A^2}\{3 - \frac{Bp[B^2+6q(q+2p)+2B(4q+p)]}{A} + \frac{3p^3B^2q^2(5B+8q+4p)}{A^2} - \frac{24p^5q^4B^3}{A^3}\},$$

$$\frac{\partial^3 T^{ij}}{\partial q^3} = \frac{\partial^3 T^{ij}}{\partial q^3} = \frac{24CB}{A^2}\{1 - \frac{2p^2B(p+4q)}{A} + \frac{3p^4B^2q(p+2q)}{A^2} - \frac{8p^6q^3B^3}{A^3}\}$$

According to (10), each of the found partial derivatives (12) was examined for the presence of zero contours within the found bifurcation space (Fig. 2), their topology in the space $X(p,q)$ and their signature were determined. The obtained results were plotted on the phase diagram of the studied system. In order to find the zero contours of the determinant of the matrix of the fourth partial derivatives of the components of T^{ij} along the $X(p,q)$ coordinates, its analytical expression was obtained. According to the obtained analytical expression of the determinant of the matrix of the fourth partial derivative components of the energy-momentum tensor of the studied island system (1) - (3), its zero contours were found within the previously found bifurcation space, the corresponding topology and signature were investigated. The obtained results were plotted on the phase diagram of the existence of the studied system. The results of calculations of the position of the spaces of the phase diagram, in which the conditions for the occurrence of at least two coexisting phases are met, are shown in Fig. 3:

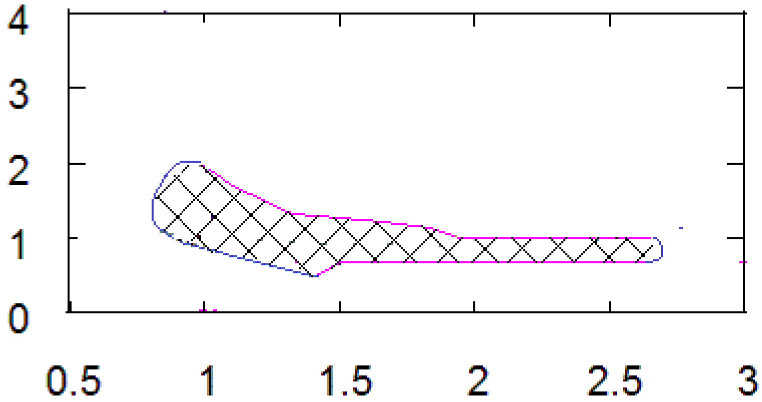

Fig. 3 - The space of the phase diagram in which the condition for the change of a stable state to another stable state for T^{11} and T^{22} is fulfilled.

Conclusions.

According to the provisions of Thom's theory of catastrophes and the generalization of the theories of Ginsburg-Landau phase transitions using the differential topological approach, it was found that the island system that corresponds to the Plebański-Demiański metric has spaces on the phase diagram of existence in which the conditions for the emergence of a stable phase, bifurcation space and space of coexistence of at least two phases at the same time. Based on the results of calculations of zero contours and the signature of the higher derivative components of the energy-momentum tensor, the positions of the spaces on the phase diagram in the coordinates of the Plebański-Demiański metric, where the conditions for the existence of a stable phase of the system are fulfilled, were obtained. The locations of the set of points of the phase diagram where the conditions for the occurrence of the bifurcation space and the space of phase coexistence of order two are fulfilled are calculated. Based on the obtained results of modeling the positions of spaces of coexistence of phases of order two, it can be assumed that the model presented in the work can be used to study the possibility of the emergence of spaces of coexistence of phases of higher orders. A more detailed analysis of the simulation results as applied to specific cosmological objects is associated with specifying specific values of the constants included in expressions (3) and (4). In work (1), these constants correspond to the mass of the object under study, NUT parameter, angular momentum per unit mass, acceleration, electric and magnetic charge. The method of analyzing the phase states of space-time can be used for both macro-objects and micro-objects in cosmology.

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