KAPITEL 5 / *CHAPTER 5* [5]
EXACT EQUATIONS OF MOTION OF TURBULENT LIQUID FLOW

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Introduction

Turbulent and laminar flows have a great influence on natural processes, as well as on energy and transport. To effectively use the capabilities of these processes, it is necessary to have the most accurate mathematical description of the movement of continuous media. This description is based on the law of conservation of momentum and uses partial differential equations. The most common flow regime is turbulent, which is characterized by the presence of three types of motion - translational, rotational and oscillatory. Currently, mathematical models are used, implemented in the form of a combination of exact and semi-empirical equations that do not take into account rotational motion [1–4]. In this paper, problems are considered within the framework of an averaged turbulence model, which takes into account the influence of translational and rotational motion [1, 2, 4].

The laws of motion of working bodies as a continuous medium are studied in fluid mechanics, the mathematical basis of which is the equations of motion in stresses (Navier).

In the classical literature, there are two exact three-dimensional special cases of these equations: the equation of motion for a rigid body (elasticity theory) and the Navier-Stokes equation for a liquid. The theory of elasticity has proven the high quality of calculations and has a clear structure of equations [5, 6]. In fluid mechanics there is no similar structure for all flow regimes, which led to a large share of experimentation and, accordingly, to a high labor intensity of research.

Writing accurate equations requires accepting the point of view that the general equation of motion must describe the most general (turbulent) flow regime.

The implementation of this point of view became possible by applying the operation of extracting the rotor of velocity from the gradient of velocity and from the Laplace operator from velocity to the Navier and Navier-Stokes equations. In this case,

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the second form of the equation was used for the total acceleration of a liquid particle in the Gromeka-Lamb form, which includes the angular velocity of rotation of the particles [4].

The equations are derived for continuous media in which shear stresses are described by the velocity gradient function - two models of a Newtonian fluid and one model of a non-Newtonian fluid with a power rheological law.

Thus, the main task of the derivation was to find the term characterizing the influence of the viscous friction force on the turbulent flow regime.

In any version of the derivation, the initial equation is the motion of a continuous medium in stresses.

$$
\begin{aligned}
X+\frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)&=\frac{du_x}{dt}\\
Y+\frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)&=\frac{du_y}{dt}\\
Z+\frac{1}{\rho}\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)&=\frac{du_z}{dt},
\end{aligned}
\tag{1}
$$

where p_{xx}, p_{yy}, p_{zz} are normal stresses, $\tau_{yx}, \tau_{zx}, \tau_{yz}$ are shear stresses, X, Y, Z - specific mass force, u_x, u_y, u_z - velocity projections, t - time.

The search for general integrals and partial solutions was carried out for two problems: flow around a horizontal plate and flow in a straight circular pipe in turbulent and laminar flow regimes. Particular solutions were compared with known analytical and semi-empirical equations.

The differential equations in this work were obtained for an incompressible fluid; the influence of pulsations of all thermodynamic quantities was not taken into account.

5.1. Non-Newtonian fluid

Let us apply system (1) to derive the equation of motion of one group of Non-Newtonian liquids (Ostwald de Waele model), which has a power-law rheological

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equation $\tau = k \cdot \varepsilon^n$, (where $\varepsilon = grad\ u$, $k = const$, $n = const$) [7, 8].

Transformation conditions:

1. Normal stress p_{ii} and pressure p_i have opposite directions.

$p_{xx} = -p_x$, $p_{yy} = -p_y$, $p_{zz} = -p_z$ (where are $p_x, p_y, p_z -$ the pressure projections).

2. Let's accept meaning constants $k = \mu$, (where μ is the dynamic viscosity of the liquid).

3. Let us express the total acceleration of a liquid particle using Gromek -Lamb equation.

This form is equivalent to the standard formula, but allows one to establish the influence of linear and angular velocity on the total acceleration of a fluid particle (du/dt).

In vector form, this equation looks like:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + grad\left(\frac{u^2}{2}\right) + 2 \cdot [\vec{\omega} \times \vec{u}]. \tag{2}$$

In projections on the coordinate axes:

$$\frac{du_x}{dt} = \frac{\partial u_x}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) + 2 \cdot (u_z\omega_y - u_y\omega_z)$$

$$\frac{du_y}{dt} = \frac{\partial u_y}{\partial t} + \frac{\partial}{\partial y}\left(\frac{u^2}{2}\right) + 2 \cdot (u_x\omega_z - u_z\omega_x)$$

$$\frac{du_z}{dt} = \frac{\partial u_z}{\partial t} + \frac{\partial}{\partial z}\left(\frac{u^2}{2}\right) + 2 \cdot (u_y\omega_x - u_x\omega_y).$$

The convective part of the total acceleration in (2) also follows from the formula vector analysis $(\vec{u} \cdot \nabla)\vec{u} = grad\left(u^2 / 2\right) + rot\ \vec{u} \times \vec{u}$

4. Let us transform the terms characterizing viscous friction using the X axis as an example (1).

Let's add zero to the derivatives inside the brackets, presenting it in the form of two specially selected terms with opposite signs.

Then we get:

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+\nu\left[\frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}-\frac{\partial u_x}{\partial y}+\frac{\partial u_x}{\partial y}\right)^n+\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}-\frac{\partial u_z}{\partial x}+\frac{\partial u_z}{\partial x}\right)^n\right]=$$

$$=\frac{\partial u_x}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)+2\left(u_z\omega_y-u_y\omega_z\right).$$

Or

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+2^n\cdot\nu\cdot\left[\frac{\partial}{\partial y}\left(\omega_z+\frac{\partial u_x}{\partial y}\right)^n+\frac{\partial}{\partial z}\left(\omega_y+\frac{\partial u_z}{\partial x}\right)^n\right]-\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=\frac{\partial u_x}{\partial t}++2\left(u_z\omega_y-u_y\omega_z\right).$$

Where - for tangential stresses [1, 2, 4]:

$$\tau_{yx}=\mu\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right),$$

$$\tau_{zx}=\mu\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right),$$

$$\tau_{yz}=\mu\left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right)$$

- for the speed rotor:

$$2\omega_y=(rot\ u)_y=\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x},$$

$$2\omega_z=(rot\ u)_z=\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}$$

$$2\omega_x=(rot\ u)_x=\frac{\partial u_z}{\partial y}-\frac{\partial u_y}{\partial z}$$

After similar transformations for other coordinate axes we get :

$$G-\frac{1}{\rho}divp+2^n\cdot\nu\cdot f(u,\omega,n)-grad\left(\frac{u^2}{2}\right)=\frac{\partial u}{\partial t}+2\left[\omega\times u\right], \quad (3)$$

where the projections of the friction function $f(u,\omega,n)$ have the form:

$$f_x(u,\omega,n)=\frac{\partial}{\partial y}\left(\omega_z+\frac{\partial u_x}{\partial y}\right)^n+\frac{\partial}{\partial z}\left(\omega_y+\frac{\partial u_z}{\partial x}\right)^n$$

$$f_y(u,\omega,n)=\frac{\partial}{\partial z}\left(\omega_x+\frac{\partial u_y}{\partial z}\right)^n+\frac{\partial}{\partial x}\left(\omega_z+\frac{\partial u_x}{\partial y}\right)^n \quad (4)$$

$$f_z(u,\omega,n)=\frac{\partial}{\partial x}\left(\omega_y+\frac{\partial u_z}{\partial x}\right)^n+\frac{\partial}{\partial y}\left(\omega_x+\frac{\partial u_y}{\partial z}\right)^n$$

Accounting influence linear (u) and angular speed (ω) rotation particles indicates that equation (3) describes turbulent regime within the framework of the averaged turbulence model.

5.1.1. Analysis of the equation of motion (3)

Equation (3) has three special cases:

1. $n = 1$.

This special case describes the flow of a Newtonian fluid with a rheological equation $\tau = \mu \cdot grad\, u$ and has the form:

$$G - \frac{1}{\rho} divp + 2 \cdot \nu \cdot f(u, \omega) - grad\left(\frac{u^2}{2}\right) = \frac{\partial u}{\partial t} + 2[\omega \times u]. \tag{5}$$

This equation was obtained in a different way in [9].

Equation (5) breaks down into two special cases:

- for laminar flow regime *(ω = 0*).

$$G - \frac{1}{\rho} divp + 2 \cdot \nu \cdot f(u) - grad\left(\frac{u^2}{2}\right) = \frac{\partial u}{\partial t}. \tag{6}$$

- for the vortex flow regime ($u = 0$).

$$G - \frac{1}{\rho} divp + 2 \cdot \nu \cdot f(\omega) = \frac{\partial(\omega \cdot r)}{\partial t}.$$

2. $\omega = 0$.

After eliminating the angular velocity ω from (3), we obtain:

$$G - \frac{1}{\rho} divp + 2^n \cdot \nu \cdot f(u, n) - grad\left(\frac{u^2}{2}\right) = \frac{\partial u}{\partial t}.$$

By definition (6) describes the laminar flow regime of a Non-Newtonian fluid.

3. $u = 0$.

After eliminating the linear velocity u from (3), we obtain:

$$G - \frac{1}{\rho} divp + 2^n \cdot \nu \cdot f(\omega, n) = \frac{\partial(\omega \cdot r)}{\partial t}.$$

This equation describes the vortex flow regime with a fixed axis of rotation.

In Fig. 1 shows the place of these equations among other equations of motion of a continuous medium.

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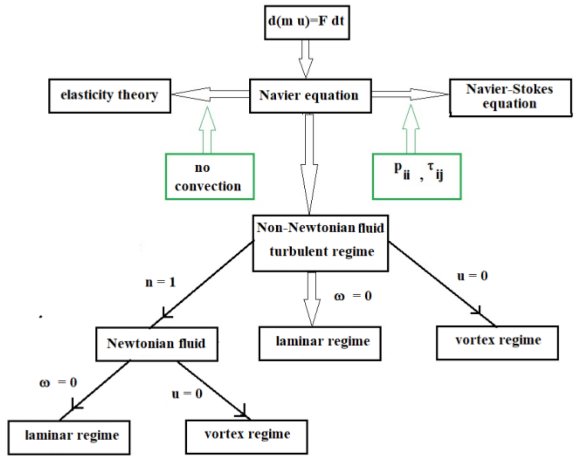

Fig. 1. Connections between various equations of motion of a continuous medium.

In the fig. 1 shows two versions of the Newtonian fluid, which differ in the system of assumptions. When deriving the Navier-Stokes equation, restrictions are imposed on the tangential (τ_{ij}) and normal stresses (For example $p_{xx} = -p + 2 \cdot \mu \cdot du_x / dx$) [4]. Equation (5) describes flows in which the same Newton’s rheological law is used, but normal stresses (pressure) can change arbitrarily.

The equation of motion (3) breaks down into two special cases when angular or linear velocity is excluded. This indicates the existence of a third flow regime, the dynamics of which depends on the particle rotation speed ω .

5.2. Newtonian fluid

5.2.1. Turbulent flow regime

Let us find the simplest special cases of equations (1, 5).

We will use two methods for finding the integrals for the velocity distribution on the plate and in the round pipe $u_x(y)$ and $u_z(r)$ [10].

1. Let's find a special case (1) for a one-dimensional flow, and then use Newton's rheological law.

2. Let’s use (5), which in coordinate form has the form:

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+2\nu\left[\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_z}{\partial x\partial z}+\frac{\partial \omega_z}{\partial y}+\frac{\partial \omega_y}{\partial z}\right]=\frac{du_x}{dt},$$

$$Y-\frac{1}{\rho}\frac{\partial p_y}{\partial y}+2\nu\left[\frac{\partial^2 u_y}{\partial z^2}+\frac{\partial u_x}{\partial x\partial y}+\frac{\partial \omega_x}{\partial z}+\frac{\partial \omega_z}{\partial x}\right]=\frac{du_y}{dt}, \quad (7)$$

$$Z-\frac{1}{\rho}\frac{\partial p_z}{\partial z}+2\nu\left[\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_y}{\partial y\partial z}+\frac{\partial \omega_y}{\partial x}+\frac{\partial \omega_x}{\partial y}\right]=\frac{du_z}{dt}.$$

Let us find a one-dimensional special case (7) and integrate it.

Flow on a horizontal plate

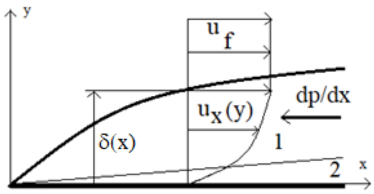

Fig. 2. Calculated flow diagram on a plate (1 – turbulent boundary layer, 2 – laminar sublayer).

Let us apply the Navier equation (1) to find the distribution of shear stress in the boundary layer (Fig. 2). For the *x* coordinate we have:

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+\frac{1}{\rho}\left(\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)=\frac{du_x}{dt}.$$

After simplification for one-dimensional and steady flow we obtain:

$$\frac{dp_x}{dx}=\frac{d\tau_{yx}}{dy}.$$

After integration with $dp_x/dx=const$ we find

$$\tau_x(y)=\frac{dp_x}{dx}y+c_1. \quad (8)$$

We find the velocity distribution normal to the surface of the plate using Newton's equation for viscous friction:

$$\tau_x(y) = \frac{dp_x}{dx} y + c_1 = \mu \frac{du_x}{dy}$$

after integration

$$u_x(y) = \frac{1}{2\mu} \frac{dp_x}{dx} y^2 + c_1 y + c_2. \tag{9}$$

Let us use the second method of finding the velocity distribution using (7) and taking into account the rotation of particles. Because speed u_x is changing only along the y axis:

$$2\mu\left(\frac{\partial \omega_z}{\partial y} + \frac{\partial^2 u_x}{\partial y^2}\right) = \frac{\partial p_x}{\partial x}.$$

By calculating the function ω_z you can get:

$$\frac{\partial^2 u_x}{\partial y^2} = \frac{1}{\mu} \frac{\partial p_x}{\partial x}. \tag{10}$$

Integrating (10) with $(1/\mu) \cdot dp_x/dx = const$ we find the general integral, which coincides with (9).

Flow in a round pipe

Let us use the general equation of motion of a continuous medium in stresses in coordinates ($r, \theta,$ z). For the z coordinate, the general equation has the form [2, 4]:

$$Z - \frac{1}{\rho} \frac{\partial p_z}{\partial z} + \frac{1}{\rho}\left(\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\tau_{zr}}{r}\right) = \frac{du_z}{dt}, \tag{11}$$

where $p_z = -p_{zz}$ – pressure along the z *axis*, which, according to the sign rule, is opposite to the normal stress p_{zz}.

Let us simplify (11), assuming that there are no mass forces and rotation around the pipe axis.

Then we get:

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{\tau_{zr}}{r} = \frac{\partial p_z}{\partial z}.$$

With a constant pipe diameter $dp_z/dz = const$, the solution has the form:

$$\tau_{zr} = \frac{c_1}{r} + \frac{{dp_z}/{dz} \cdot r}{2}.$$

Because

$$\tau_{zr} = \mu \frac{du_z}{dr} = \frac{1}{2} \frac{dp_z}{dz} r + \frac{c_1}{r},$$

then after integration we get:

$$u_z(r) = \frac{1}{4\mu} \frac{dp_z}{dz} r^2 + c_1 \ln r + c_2. \tag{12}$$

Let us use the second solution method, for which (7) in a cylindrical coordinate system for the *z axis* takes the form:

$$Z - \frac{1}{\rho} \frac{\partial p_z}{\partial z} + 2\nu \left(\frac{\partial}{\partial r} \left[\omega_\theta + \frac{\partial u_z}{\partial r} \right] + \frac{1}{r} \frac{\partial}{\partial \theta} \left[\omega_r + \frac{\partial u_\theta}{\partial z} \right] + \frac{1}{r} \left[(\omega_\theta + \frac{\partial u_z}{\partial r} \right] \right) = \frac{du_z}{dt}.$$

For this problem, the flow velocity changes only along the radius and the derivative along the z axis can be neglected. We will also assume that the speed does not change depending on the angle θ (the flow does not rotate around the pipe axis).

Then, calculating ω_r and making reductions, we find:

$$\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} = \frac{1}{\mu} \frac{\partial p_z}{\partial z}. \tag{13}$$

After integrating (13) with $(1/\mu) \cdot dp_z / dz = const$ we again obtain (12).

Thus, the general integral for the velocity distribution in a turbulent regime is equation (12).

In Fig. 3 shows a diagram for obtaining general integrals for finding tangential stresses and velocity distribution for a turbulent flow regime.

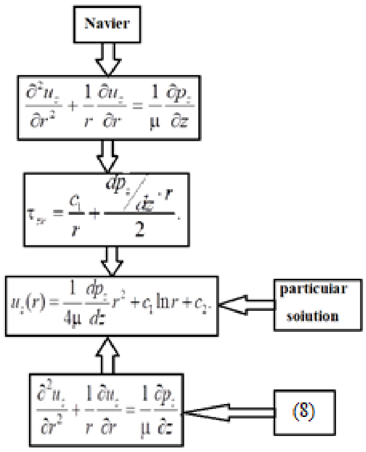

Fig. 3. Two ways to find the general integral for turbulent flow in a round pipe.

Table 1 - General integrals for turbulent flow

Mode	Pipe flow	Flow on the plate
Turbulent	$u_z(r)=\frac{1}{4\mu}\frac{dp_z}{dz}r^2+c_1\ln r+c_2$	$u_x(y)=\frac{1}{2\mu}\frac{dp_x}{dx}y^2+c_1y+c_2$

Private solutions

1. We will find a particular solution for the velocity distribution in the pipe from the general solution (12) using the variable y (Fig. 4).

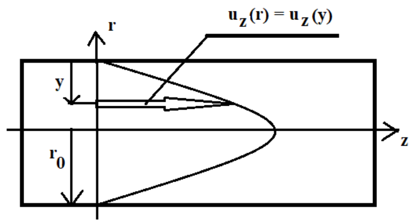

Fig. 4. Calculation scheme for searching for a particular solution ($y = r_0 - r$).

Let us apply to these equations the boundary conditions characteristic of the pipe axis:

for $y= r_0$, $\tau=0$ and $u_z(y)= u_{max}$.

Then we get:

$$u_z(y)=\frac{1}{4\mu}\frac{dp_z}{dz}\left(y^2-r_0^2+2r_0^2\ln\frac{r_0}{y}\right)+u_{\max}. \qquad (14)$$

This solution is not applicable to calculating the velocity on the wall *(y = 0)* ,

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since near the wall there is a laminar sublayer that separates the core of the turbulent flow from the wall.

In Fig. 5 shows a comparison of the velocity distribution constructed according to (14) and the semi-empirical equation $u_z(y) = u_{max}(y/r_0)^{0.16}$ [4].

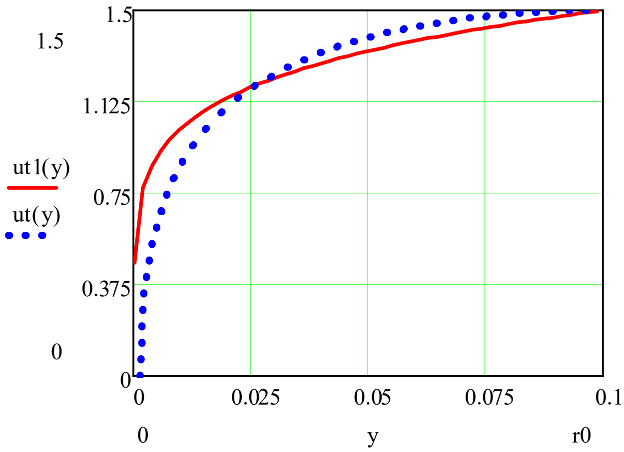

Fig. 5. Comparison of the theoretical velocity distribution in a pipe for a turbulent flow regime - solid , for a power-law semi-empirical equation - dots.

2. Let us consider a particular solution for turbulent flow on a plate. Let us set the conditions on the outer boundary of the boundary layer and apply them to the integral of Table 1.

When $y = \delta(x)$, $\tau_x(y) = 0$, and $u_x(y) = u_f$.

Then we get:

$$u_x(y) = \frac{1}{2\mu}\frac{dp_x}{dx}\left[y^2 + \delta(x)^2 - 2y \cdot \delta(x)\right] + u_f. \tag{15}$$

In Fig. 6 shows a comparison of the velocity distribution found from (16) and from the well-known semi-empirical equation $ue(y) = u_x(y) = u_f[y/\delta(x)]^{1/7}$ [4].

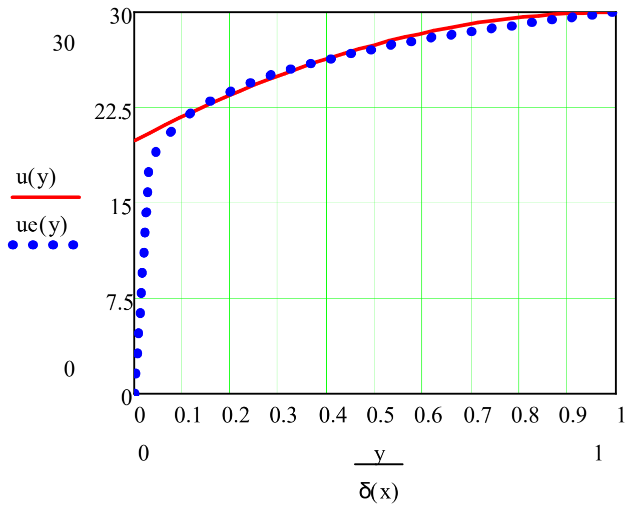

Fig. 6. Comparison of the theoretical velocity distribution for turbulent flow on a plate (15) – solid line and semi-empirical – points / *Re $_x$ =2·10 ^{5}, dp $_x$/ dx =–35 Pa/m, air*

As follows from a comparison of the curves, in the range $y/\ \delta(x) = 0...0.1$ there is a sharp discrepancy between the theoretical and semi-empirical dependence.

This is due to the presence of a laminar sublayer in which there is no rotation of particles and the velocity distribution in this range $y/\ \delta(x)$ must be calculated using a different equation.

In table 2 shows particular solutions that were obtained as a result of applying boundary conditions to the previously found general integrals.

Table 2 - Particular solutions for turbulent flow

Mode	Pipe flow	Flow on the plate
Turbulent	$u_z(y) = \frac{1}{4\mu}\frac{dp_z}{dz}\left(y^2 - r_0^2 + 2r_0^2 \ln\frac{r_0}{y}\right) + u_{\max}$	$u_x(y) = \frac{1}{2\mu}\frac{dp_x}{dx}\left[y^2 + \delta(x)^2 - 2y\cdot\delta(x)\right] + u_f$

Note: $\delta(x)$ is the thickness of the boundary layer.

5.2.2. Laminar flow regime.

Let's consider the flow on a horizontal plate and in a round pipe, find general integrals and particular solutions.

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Flow on a horizontal plate

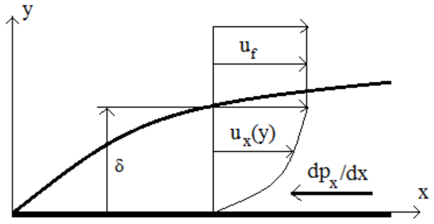

Fig. 7. Scheme of laminar flow on a plate.

To find the general integrals, we will use the well-known calculation scheme (Fig. 7) and (6) in coordinate form.

$$\begin{aligned} &X - \frac{1}{\rho}\frac{\partial p_x}{\partial x} + 2\nu\left[\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_z}{\partial x \partial z}\right] = \frac{du_x}{dt}, \\ &Y - \frac{1}{\rho}\frac{\partial p_y}{\partial y} + 2\nu\left[\frac{\partial^2 u_y}{\partial z^2} + \frac{\partial u_x}{\partial x \partial y}\right] = \frac{du_y}{dt}, \\ &Z - \frac{1}{\rho}\frac{\partial p_z}{\partial z} + 2\nu\left[\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_y}{\partial y \partial z}\right] = \frac{du_z}{dt}. \end{aligned} \tag{16}$$

Let us find a particular solution by simplifying (16) for a one-dimensional steady flow with its subsequent integration.

Then

$$\frac{d^2 u_x}{dy^2} = \frac{1}{2\mu}\frac{dp_x}{dx}$$

Let us assume that the dynamic viscosity is constant and the pressure drop along the x axis does not change $\left(\frac{1}{2\mu}\frac{dp_x}{dx} = const\right)$.

After integration:

$$u_x(y) = \frac{1}{4\mu}\frac{dp_x}{dx}y^2 + c_1 y + c_2 .$$

Private solution

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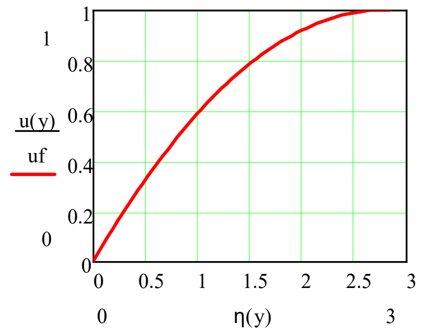
For a laminar flow regime, the boundary layer begins at the wall and the boundary conditions will be as follows: for $y=\delta(x)$, $u_x(y)=u_f$, and for $y=0$, $u_x(y)=0$. As a result, we obtain for laminar flow:

$$u_x(y)=\frac{1}{4\mu}\frac{dp_x}{dx}\left[y^2-y\cdot\delta(x)\right]+\frac{y}{\delta(x)}u_f \qquad (17)$$

Let us compare the solution (17) with the Blasius solution, which is presented in tabular form [4]. When constructing the graph, the dimensionless coordinates $u_x(y)/u_f=f\left(\eta=\frac{y}{2}\sqrt{\frac{u_f}{\upsilon\cdot x}}\right)$ adopted in [4] were used. The numerical data correspond to the laminar flow regime at a distance *x* from the leading edge.

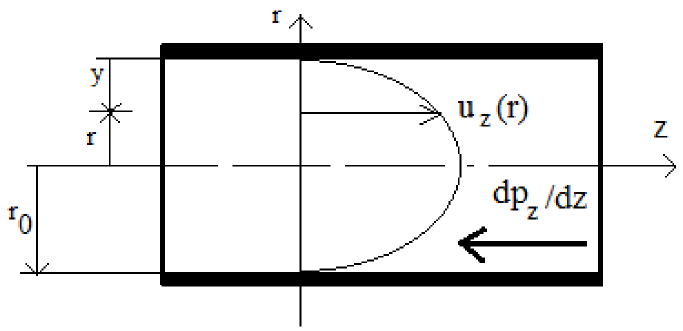
Comparison results of the graph in Fig. 8 and the table data for the Blasius solution show their satisfactory agreement.



Rice. 8. Theoretical velocity distribution over the thickness of the laminar boundary layer on the plate *(Re_x=5.8·10^3 , dp_x/dx =–3.2 Pa/m, air)* .

Flow in a round pipe

In Fig. Figure 9 shows a design diagram of a stabilized flow in a round pipe.



Rice. 9. Calculation scheme for laminar flow in a round pipe.

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A special case (16) for a one-dimensional flow in coordinates r , z has the form:

$$\frac{\partial^2 u_z}{\partial r^2} = \frac{1}{2\mu} \frac{\partial p_z}{\partial z}. \tag{18}$$

After double integration at $(1/2\ \mu)\ dp_z / dz = const$ we find:

$$u_z(r) = \frac{1}{4\mu} \frac{dp_z}{dz} \cdot r^2 + c_1 r + c_2. \tag{19}$$

Table 3 - General integrals for laminar flow regime.

Mode	Pipe flow	Flow on the plate
Laminar	$u_z(r) = \frac{1}{4\mu} \frac{dp_z}{dz} \cdot r^2 + c_1 r + c_2$	$u_x(y) = \frac{1}{4\mu} \frac{dp_x}{dx} \cdot y^2 + c_1 y + c_2$

Private solution

Let us find a particular solution for pipe (19) under the following boundary conditions:

If $r=0$, then o $du_z / dr = 0$, and when $r = r_0$, $u_z = 0$. Then we obtain the velocity distribution along the radius, which coincides with the known equation Poiseuille [1, 2, 4].

$$u_z(r) = \frac{1}{4\mu} \partial p_z / \partial z \cdot \left(r^2{}_0 - r^2\right),$$

In Fig.11 shows a diagram for deriving the general equation of motion (5) and obtaining particular solutions for flow in a pipe and on a plate.

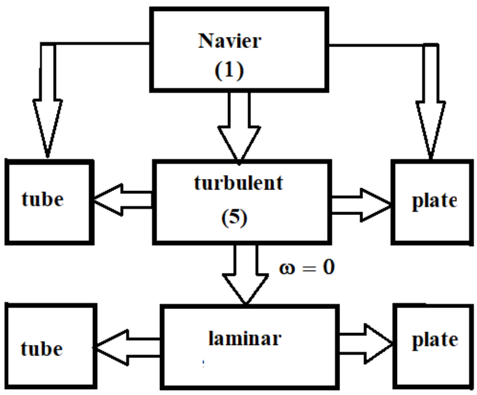

Fig. 11. Scheme for obtaining private solutions

Thus, to find exact solutions for the laminar flow regime, it is necessary to use equation (6).

5.3. Stokes liquid

5.3.1. Analysis of Assumptions

The Stokes equation is derived in two ways: using general theorems of mathematics and using the equation of motion in stresses (1) [2, 4]. Let's consider the analysis of assumptions for the second option for deriving the equation.

Table 5. Constraints for deriving equation (20).

N	Restrictions	Stress
1 .	$p_{xx} = -p + 2\mu \frac{\partial u_x}{\partial x}$	Normal
2 .	$p_{yy} = -p + 2\mu \frac{\partial u_y}{\partial y}$	Normal
3 .	$p_{zz} = -p + 2\mu \frac{\partial u_z}{\partial z}$	Normal
4 .	$\tau = \mu \cdot grad\ u$	Tangents
5 .	$\rho = const$	

From Table 5 it follows that the restrictions in the Stokes equation apply to all components of the surface force. This means that normal and shear stresses must change according to certain rules. (In equation (5), normal stresses can change arbitrarily.)

Let us analyze the Stokes equation for an incompressible fluid:

$$G - \frac{1}{\rho} grad\ p + \nu \cdot \nabla^2 u = \frac{du}{dt}, \tag{20}$$

From (20) it follows that the main factor taking into account the dynamics of the flow is the linear velocity, which affects the terms of the force of viscous friction and inertia.

5.3.2. Second form and special cases

Let us reduce the Stokes equation to a form more convenient for subsequent analysis and combine the terms that take into account viscosity. Let's perform the transformation using the *x* coordinate as an example.

– From the limit for normal stress (Table 5):

$$p_{xx} = -p + 2\mu \frac{\partial u_x}{\partial x} = -p_x \quad \text{or} \quad p = p_x + 2\mu \frac{\partial u_x}{\partial x},$$

where normal stress $p_{xx} = -p_x$ according to the sign rule.

Then the pressure term will take the form:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\frac{\partial}{\partial x}\left(p_x + 2\mu \frac{\partial u_x}{\partial x}\right) = -\frac{1}{\rho}\frac{\partial p_x}{\partial x} - 2\nu \frac{\partial^2 u_x}{\partial x^2}. \tag{21}$$

– Let us transform the Laplace operator and select terms that take into account the influence of linear and angular velocity.

Then $$\nabla^2 u_x = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}.$$

Let's express the second and third terms in terms of the first derivative, add a zero in parentheses and present it in the form of two identical terms with different signs.

$$\frac{\partial^2 u_x}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} + \frac{\partial u_y}{\partial x}\right) = \frac{\partial^2 u_y}{\partial x \partial y} - \frac{\partial (rot\ u)_z}{\partial y} = \frac{\partial^2 u_y}{\partial x \partial y} - 2\frac{\partial \omega_z}{\partial y}$$

$$\frac{\partial^2 u_x}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} + \frac{\partial u_z}{\partial x}\right) = \frac{\partial^2 u_z}{\partial x \partial z} + \frac{\partial (rot\ u)_y}{\partial z} = \frac{\partial^2 u_z}{\partial x \partial z} + 2\frac{\partial \omega_y}{\partial z}.$$

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From these equations it follows that there is a function $\psi(u, \omega)$, which depends on two arguments and has a component on the x axis in the form:

$$\psi_x(u,\omega) = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z} + 2\left(\frac{\partial \omega_y}{\partial z} - \frac{\partial \omega_z}{\partial y}\right).$$

Taking into account the last equation and (21), we obtain:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \cdot \nabla^2 u_x = -\frac{1}{\rho}\frac{\partial p_x}{\partial x} + \nu\left[-\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z} + 2\left(\frac{\partial \omega_y}{\partial z} - \frac{\partial \omega_z}{\partial y}\right)\right].$$

Carrying out similar transformations for the y and z axes we get:

$$\begin{aligned}
\varphi_x(u,\omega) &= -\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z} - 2\left(rot\ \omega\right)_x \\
\varphi_y(u,\omega) &= \frac{\partial^2 u_x}{\partial x \partial y} - \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial y \partial z} - 2\left(rot\ \omega\right)_y \\
\varphi_z(u,\omega) &= \frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_y}{\partial y \partial z} - \frac{\partial^2 u_z}{\partial z^2} - 2\left(rot\ \omega\right)_z ,
\end{aligned} \tag{22}$$

Where - $\left(rot\ \omega\right)_x = \frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z};\ \left(rot\ \omega\right)_y = \frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x};\ \left(rot\ \omega\right)_z = \frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y}.$

Taking into account (22) and the total derivative in form (2), the Stokes equation can be written:

$$\begin{aligned}
&X - \frac{1}{\rho}\frac{\partial p_x}{\partial x} + \nu \cdot \left[-\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z} - 2\left(rot\ \omega\right)_x\right] - \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = \frac{\partial u_x}{\partial t} + 2(u_z\omega_y - u_y\omega_z) \\
&Y - \frac{1}{\rho}\frac{\partial p_x}{\partial y} + \nu \cdot \left[\frac{\partial^2 u_x}{\partial x \partial y} - \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial y \partial z} - 2\left(rot\ \omega\right)_y\right] - \frac{\partial}{\partial y}\left(\frac{u^2}{2}\right) = \frac{\partial u_y}{\partial t} + 2(u_x\omega_z - u_z\omega_x) \\
&Z - \frac{1}{\rho}\frac{\partial p_z}{\partial z} + \nu \cdot \left[\frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_y}{\partial y \partial z} - \frac{\partial^2 u_z}{\partial z^2} - 2\left(rot\ \omega\right)_z\right] - \frac{\partial}{\partial z}\left(\frac{u^2}{2}\right) = \frac{\partial u_z}{\partial t} + 2(u_y\omega_x - u_x\omega_y).
\end{aligned} \tag{23}$$

In this form of recording, the terms taking into account viscous friction and inertia have the same influencing factors - (u, ω). In brief form, system (23) can be written:

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$$G-\frac{1}{\rho}div\ p+\nu\cdot\varphi(u,\omega)-grad\left(\frac{u^2}{2}\right)=\frac{\partial u}{\partial t}+2\left[\vec{\omega}\times\vec{u}\right]. \qquad (24)$$

From (24) it follows that the Stokes equation describes the turbulent flow regime within the framework of an averaged model.

The derivation of this equation was carried out without the use of additional restrictions. This means that (24) can be considered the second form of writing the Stokes equation (20).

Let's consider some special cases (24).

1. By eliminating viscosity ($\nu = 0$) , we obtain a general equation for inviscid flow from which we can obtain the Bernoulli equation for an ideal fluid:

$$G-\frac{1}{\rho}div\ p-grad\left(\frac{u^2}{2}\right)=\frac{\partial u}{\partial t}+2\left[\vec{\omega}\times\vec{u}\right]. \qquad (25)$$

2. For a laminar flow regime, the angular velocity $\omega(x,y,z)=0$ and equation (24) will take the form:

$$\begin{aligned}
&X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+\nu\cdot\left(-\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_y}{\partial x\partial y}+\frac{\partial^2 u_z}{\partial x\partial z}\right)-\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=\frac{\partial u_x}{\partial t}\\
&Y-\frac{1}{\rho}\frac{\partial p_y}{\partial y}+\nu\cdot\left(\frac{\partial^2 u_x}{\partial x\partial y}-\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_z}{\partial y\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)=\frac{\partial u_y}{\partial t} \qquad (26)\\
&Z-\frac{1}{\rho}\frac{\partial p_z}{\partial z}+\nu\cdot\left(\frac{\partial^2 u_x}{\partial x\partial z}+\frac{\partial^2 u_y}{\partial y\partial z}-\frac{\partial^2 u_z}{\partial z^2}\right)-\frac{\partial}{\partial z}\left(\frac{u^2}{2}\right)=\frac{\partial u_z}{\partial t}.
\end{aligned}$$

In short form, system (26) has the form:

$$G-\frac{1}{\rho}div\ p+\nu\cdot\varphi(u)-grad\left(\frac{u^2}{2}\right)=\frac{\partial u}{\partial t}. \qquad (27)$$

3. When $u(x,y,z)=0$ from (24) we obtain the equation for a standing vortex

$$G-\frac{1}{\rho}div\ p-\nu\cdot\varphi(\omega)=\frac{\partial(\omega r)}{\partial t}.$$

Particular problems

Let us find the integrals for the cases of flow on a horizontal plate and in a straight circular pipe.

Flow on a horizontal plate

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The goal of solving the problem is to find the velocity distribution normal to the surface.

Finding solutions is done in two ways. The first method uses the Navier equation (1), and the second uses the Stokes equation (20).

Using the first method, the shear stress distribution is found, and then the velocity distribution is found using Newton's law for viscous friction. Using the second method, the velocity distribution is found by integrating the one-dimensional second-order equation of motion. Both methods complement each other and should give the same result.

The first method was implemented earlier (section 2.1) and as a result the integral (9) was obtained.

Let us use the Stokes equation in the form (20), which after simplification takes the form:

$$\frac{d^2u_x}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}.$$

After double integration, we obtain the equation of motion for turbulent flow (9):

$$u_x(y) = \frac{1}{2\mu}\frac{dp}{dx}y^2 + c_1 y + c_2.$$

In Fig.12 shows a diagram for finding general integrals for the distribution of tangential stress and velocity during flow on a plate.

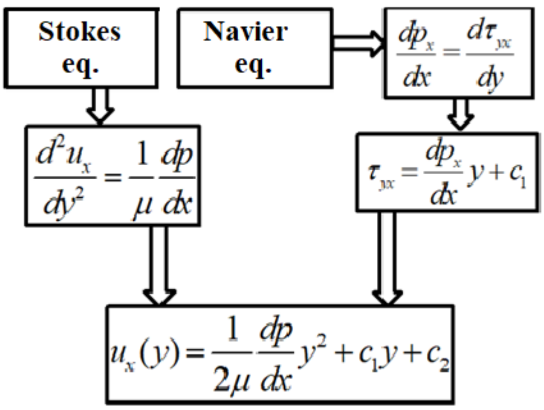

Fig. 12. Scheme for finding integrals for turbulent flow on a plate.

To find the velocity distribution in the laminar sublayer, it is necessary to use equation (27), but there is no term d^2u_x/dy^2. This means that it is impossible to find the velocity distribution for laminar flow.

Let us find a particular solution to equation (9), excluding the assumption that the flow sticking hypothesis on the wall is fulfilled. Let us use the boundary conditions only on the outer boundary of the boundary layer: for $y = \delta(x)$, $\tau_x(y)=0$, and $u_x(y = \delta) = u_f$.

Then we get:

$$u_x(y) = \frac{1}{2\mu}\frac{dp_x}{dx}\left[y^2 + \delta(x)^2 - 2y\cdot\delta(x)\right] + u_f. \tag{28}$$

Equation (28) coincides with (15).

Flow in a round pipe

Let us consider the flow in a straight circular pipe and find the general integrals for the distribution of tangential stress and velocity along the radius of the pipe (Fig. 13).

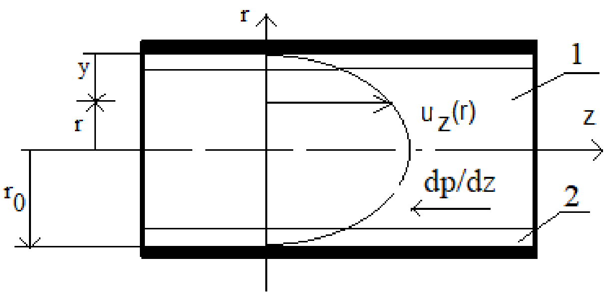

Fig. 13. Calculation scheme of flow in a pipe (1-turbulent core, 2-laminar sublayer)

Let us use the Stokes equation in the form (20) in (r, z) coordinates. Since the flow is steady and one-dimensional, the equation has the form:

$$\frac{d^2u_z}{dr^2} + \frac{1}{r}\frac{du_z}{dr} = \frac{1}{\mu}\frac{dp}{dz}.$$

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After double integration we get $\left[(1/\mu)(dp/dz)=const\right]$:

$$u_z(r)=\frac{1}{4\mu}\frac{dp}{dz}r^2+c_1\ln r+c_2. \tag{29}$$

Let's solve the same problem using the Navier equation .

From the equation (1):

$$Z-\frac{1}{\rho}\frac{\partial p_z}{\partial z}+\frac{1}{\rho}\left(\frac{\partial\tau_{zr}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta}+\frac{\tau_{zr}}{r}\right)=\frac{du_z}{dt}, \tag{30}$$

where p_z – pressure along the z axis, which, according to the sign rule, is opposite to the normal stress p_{zz}.

Let us simplify (30), assuming that there are no mass forces and rotation of the flow around the pipe axis.

Then we get:

$$\frac{\partial\tau_{zr}}{\partial r}+\frac{\tau_{zr}}{r}=\frac{\partial p_z}{\partial z}. \tag{31}$$

At a constant pipe diameter ($dp_z/dz = const$), solution (31) has the form:

$$\tau_{zr}=\frac{c_1}{r}+\frac{{dp_z}/{dz}\cdot r}{2}.$$

Let's use Newton's equation $\tau_{zr}=\mu\cdot\frac{du_z}{dr}$:

$$\mu\frac{du_z}{dr}=\frac{1}{2}\frac{dp_z}{dz}r+\frac{c_1}{r}.$$

After integration we obtain equation (29).

In Fig. 14 shows a diagram for finding integral (29) in two ways.

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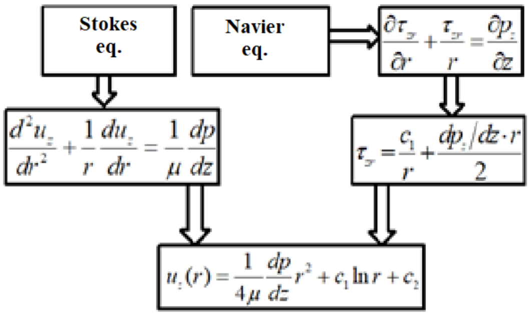

Fig. 14. Scheme for finding the integral for turbulent flow in a pipe.

Private solutions

Let's find the quotient solution equation (29) for the following boundary conditions : at $y = r_0$, $\tau = 0$ and $u_z = u_{max}$.

Then we get:

$$u_z(y) = \frac{1}{4\mu}\frac{dp_z}{dz}\left(y^2 - r_0^{\,2} + 2r_0^{\,2}\ln\frac{r_0}{y}\right) + u_{\max}. \tag{32}$$

The result coincides with (14).

Thus, particular solutions of the Stokes equation coincide with similar solutions for a Newtonian fluid in a turbulent flow regime.

Laminar flow regime

For the laminar flow regime there are no derivatives $\frac{\partial^2 u_x}{\partial y^2}$ and $\frac{\partial^2 u_z}{\partial r^2}$ and it is not possible to find the speed distribution.

This property can be interpreted as the absence of a laminar flow regime.

Thus, in a Stokes fluid only a turbulent regime is possible and the velocity distribution is described by equation (14, 15) and Fig. 5, 6. A similar nature of the velocity distribution is observed in rarefied gas (Fig. 15 for different Knudsen numbers (Kn)) [11].

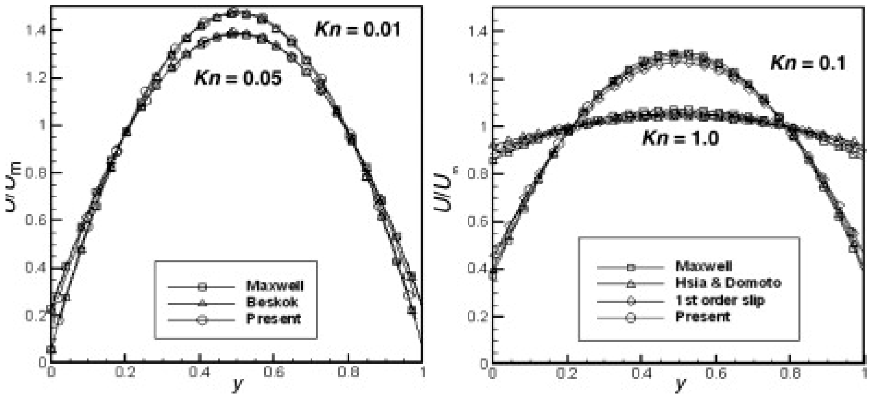

Fig. 15. Velocity distribution in a rarefied gas flow in a channel.

To determine the compliance/inconsistency of equations (14, 15) with this flow property, it is necessary to conduct additional experiments.

In Fig. 16 shows a diagram for transforming the classical version of the Stokes equation for an incompressible fluid and obtaining partial solutions.

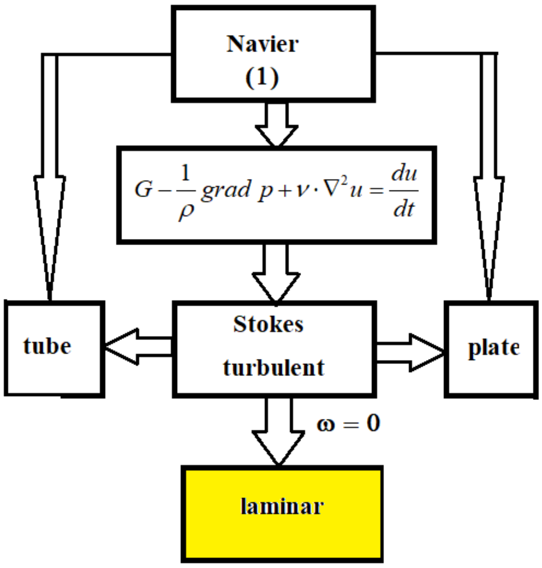

Fig. 16. Scheme for obtaining partial solutions of the Stokes equation

From Fig. 16 it follows that the equations of motion for particular problems can be obtained in two ways.

Conclusion

The three-dimensional equations found use the same system of forces and have the same structure. A comparison of these equations for various models of turbulent flow shows that they differ in only one term - for viscous friction.

The breakdown of these equations according to physically known conditions allows us to assume that there are three flow regimes - turbulent, laminar and vortex. In Fig. 17 shows an example of the third flow regime over the surface of boiling water.

Fig. 17. Vortex tube compressing a cloud of water vapor.

Despite the large amount of information, the properties of the vortex tube have not been sufficiently studied and require research from low to the maximum possible rotation speeds [12, 13].

Further progress in energy is associated with the theoretical and experimental study of the third (vortex) flow mode, which has an increased energy density. The mechanical properties of this flow regime make it possible to reduce the requirements for the thermal properties of the solid wall of the channel, which can lead to an increase in the temperature of the working fluid and will favorably affect the efficiency of the heat engine.

General equations have nine or six unknowns and are not closed. In fluid mechanics there are no exact equations for solving the closure problem, but such equations exist in another area of continuum mechanics - the theory of elasticity. The role of closing equations is played by six second-order Saint-Venant equations, written in the form of deformations. This form of recording does not allow these equations to be used to describe the movement of a fluid, because The dynamics of the flow depends on the strain rates. The transformation of these equations for a fluid medium was carried out in [14], which made it possible to obtain six third-order equations, which break down into two special cases according to the same physical conditions.

Saint-Venant's equations refer to a different way of describing motion and are derived using the rules of geometry. This allows them to be used for any continuous medium.

Solving the closure problem will allow the use of general equations to improve application programs in the field of fluid dynamics. In addition, the possibility of drawing up an equation of motion for the electrolyte opens up.

The analysis of the possibilities of general equations of motion was carried out using the example of two well-known one-dimensional problems - flow around a horizontal plate (external problem of hydrodynamics) and flow in a straight round pipe (internal problem of hydrodynamics). General integrals and some particular solutions are found.

A comparison of these solutions with known data showed that for a laminar flow regime the velocity distribution coincides with the Blasius solution for a plate and with the Poiseuille equation for flow in a round pipe. The velocity distribution for the turbulent flow regime is in satisfactory agreement with semi-empirical equations, but requires additional experimental verification.

Integrals for the turbulent regime were obtained in two ways:

1. From the equation of motion in stresses (Navier) taking into account Newton’s rheological equation.

2. By simplifying three-dimensional equations for various fluid models and integrating special cases. Both paths lead to the same results.

Analysis of the Stokes equation showed that it has a second form of notation, which includes the angular velocity of particle rotation. This form of the equation breaks down into two special cases, similar to other flow models.

The search for partial solutions turned out to be possible only for the turbulent flow regime. For laminar mode, there are no differential equations for the velocity distribution on the plate and in the pipe. This property of the Stokes equation can apply to flows in which the laminar regime does not exist. This may mean that the sticking hypothesis is not satisfied for solutions of the Stokes equation and the flow velocity on the wall is not zero. This conclusion follows from the analysis of equations (28) and (32).

One of the possible variants of such a flow is the movement of a rarefied gas in the presence of particle sliding [4,11]. Testing this assumption will make it possible to clarify the physical meaning of the Stokes equation and will have a positive impact on progress in many areas of energy.

It is characteristic that special cases of the Navier equation (3, 5, 6) are more complex versions of the Bernoulli equation for an ideal fluid. The same applies to the second form of the Stokes equation (24).

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