KAPITEL 11 / *CHAPTER 11* [11]
INTERPOLATION OF OPERATORS BY NEWTON-TYPE POLYNOMIALS

DOI: 10.30890/2709-2313.2024-27-00-016

Introduction

In the theory of operator interpolation, the construction and investigation of interpolation polynomials in abstract linear spaces plays an important role, among which Newton-type operator interpolation formulas occupy a significant place.

We derive three fundamentally new constructions of interpolation operator polynomials, using the traditional technique for this problem. The first construction is obtained in subsection 1 for an abstract Banach space using a specially chosen countable sequence of nodes associated with the basis of the space. The constructed interpolation polynomials have the property of uniqueness and invariance.

The second construction is described in subsection 2 for a specific Banach space $C[0,\infty)$ using a finite sequence of continuum nodes. Here, we also construct interpolation polynomials (of integral form) that have the property of uniqueness and invariance with respect to all integral polynomials of the same degree.

In subsection 3, the problem of the existence of a unique, invariant interpolation operator polynomial in Hilbert space, defined by its values on a continuous set of nodes, is formulated and solved. For the polynomial obtained there to be interpolable, it is necessary and sufficient that the discrete analog of the substitution rule is fulfilled. However, the substitution rule imposes significant restrictions on the interpolated operator. Therefore, in subsection 4, we construct and investigate a third-degree polynomial that does not require the substitution rule.

The results of this chapter have been published in [1 – 5].

[11] **Authors:** *Demkiv Ihor Ivanovych*

 ISBN 978-3-98924-038-4

11.1. Polynomial interpolation in a Banach space with a basis

Let us assume that X,Y – linear spaces, π_n – a set of polynomials of $P_n : X \to Y$, n – degree of the form

$$\pi_n = \{P_n : P_n(x) = L_0 + L_1 x + \ldots + L_n x^n\}, \quad (1)$$

where $L_0 \in Y, \quad L_k x^k = L_k(x,\ldots x): X \to Y, k = 1,2,\ldots,n-k$ are the operator degrees obtained from symmetric k-linear operator forms $L_k(V_1,V_2,\ldots,V_k)$: $X^k \to Y$ at $V_1 = V_2 = \ldots V_k = x$. Assume that the elements $x_1,x_2,\ldots,x_m$ $(m \geq n)$ are linearly independent from X. We construct the set $Z(m) = \{z_i\}_{i=0}^N$, $N = N(m)$ as follows: $z_0 = 0$, $z_i = x_i$, $i = 1,2,\ldots,m$, and the rest $z_{m+1}, z_{m+2},\ldots,z_N$ as all possible sums (including repetitions) of elements $x_1,x_2,\ldots,x_m$ by two, three, and so on to the n terms in each. It is easy to show that $N = \sum_{k=1}^{n} C_{m+k-1}^{k}$. Let $F : X \to Y$ some, generally nonlinear, operator be given by its values $F(z_i)$, $i = 0,1,2,\ldots,N$. Then, based on [6], the following result holds.

Theorem 1. *Let the polynomial* $P_{m,n}^I(F;x) \in \pi_n$, *the values of the* k*-linear operator forms* $L_k^I(x_{i_1},x_{i_2},\ldots,x_{i_k})$, $k = 0,1,\ldots,n$, $1 \leq i_j \leq m$ *be defined by [6]*

$$L_0^I = F(0),$$

$$L_n^I(x_{i_1},x_{i_2},\ldots,x_{i_n}) = \frac{1}{n!}\{\phi(x_{i_1} + x_{i_2} + \ldots + x_{i_n}) -$$
$$-\left[\phi(x_{i_1} + x_{i_2} + \ldots + x_{i_{n-1}}) + \phi(x_{i_1} + x_{i_2} + \ldots + x_{i_{n-2}} + x_{i_n}) + \ldots +\right.$$
$$\left.+\phi(x_{i_2} + x_{i_3} + \ldots + x_{i_n})\right] + \left[\phi(x_{i_1} + x_{i_2} + \ldots + x_{i_{n-2}}) +\right.$$
$$\left.+\phi(x_{i_1} + x_{i_2} + \ldots + x_{i_{n-3}} + x_{i_n}) + \ldots + \phi(x_{i_3} + x_{i_4} + \ldots + x_{i_n})\right] + \ldots +$$
$$+(-1)^{n-1}[\varphi(x_{i_1}) + \varphi(x_{i_2}) + \ldots + \varphi(x_{i_n})], \quad (2)$$

where $\varphi(x)=F(x)-F(0)$, *and for the definition* $L_{n-1}^{I}(x_{i_1},x_{i_2},\dots,x_{i_{n-1}})$ *we need to replace in (2)* n *by* $n-1$, $\varphi(x)$ *by* $\varphi(x)-L_n^I x^n$, *for the definition* $L_{n-2}^{I}(x_{i_1},x_{i_2},\dots,x_{i_{n-2}})$ *–* n *by* $n-2$, $\varphi(x)$ *by* $\varphi(x)-L_{n-1}^I x^{n-1}-L_n^I x^n$ *etc. Then it* $P_{m,n}^I(F;x)$ *is interpolable for* F *on the set of nodes* $Z(m)$, *which means that the conditions hold*

$$P_{m,n}^I(F;z_i)=F(z_i),\quad i=0,1,2,\dots,N. \tag{3}$$

Note that the interpolant $P_{m,n}^I(F;x)$ does not have the properties of uniqueness and invariance on the set π_n. Let us explain this fact, and the construction of the polynom $P_{m,n}^I(F;x)$ using the example $m=n=2$. We note that $Z(2)=\{z_i\}_{i=0}^{5}$ $(N=5)$, $z_0=0$, $z_1=x_1$, $z_2=x_2$ (x_1,x_2 – linearly independent), $z_3=2x_1$, $z_4=2x_2$, $z_5=x_1+x_2$. From the conditions (3) we obtain

$$P_{2,2}^I(F;z_0)=L_0^I=F(0),$$

$$P_{2,2}^I(F;z_1)=L_1^I x_1+L_2^I x_1^2=\varphi(x_1),$$

$$P_{2,2}^I(F;z_2)=L_1^I x_2+L_2^I x_2^2=\varphi(x_2),$$

$$P_{2,2}^I(F;z_3)=2L_1^I x_1+4L_2^I x_1^2=\varphi(2x_1),$$

$$P_{2,2}^I(F;z_4)=2L_1^I x_2+4L_2^I x_2^2=\varphi(2x_2),$$

$$P_{2,2}^I(F;z_5)=L_1^I x_1+L_1^I x_2+L_2^I x_1^2+2L_2^I(x_1,x_2)+L_2^I x_2^2=\varphi(x_1+x_2).$$

From this system of linear equations we find

$$L_2^I(x_1,x_2)=\frac{1}{2}\{\varphi(x_1+x_2)-\varphi(x_1)-\varphi(x_2)\},\qquad L_2^I x_1^2=\frac{1}{2}\{\varphi(2x_1)-2\varphi(x_1)\},$$

$$L_2^I x_2^2=\frac{1}{2}\{\varphi(2x_2)-2\varphi(x_2)\},\qquad L_1^I x_1=\frac{1}{2}\{4\varphi(x_1)-\varphi(2x_1)\},$$

$$L_1^I x_2=\frac{1}{2}\{4\varphi(x_2)-\varphi(2x_2)\},\qquad L_0^I=F(0), \tag{4}$$

which is consistent with formulas (2). As mentioned above, the interpolant $P_{2,2}^I(F;x)$

 ISBN 978-3-98924-038-4

with the found values of k-linear forms is not unique and invariant on the set of polynomials of the second degree. For example,

$$L_0^I = K_0^I \in R_1,\quad L_1^I x = \int_0^1 K_1(t)x(t)dt,\quad L_2^I x^2 = \int_0^1\int_0^1 K_2(t_1,t_2)x(t_1)x(t_2)dt_1dt_2 .$$

Then according to (4) we obtain

$$K_0^I = F(0),\quad \int_0^1 K_1^I(t)x_i(t)dt = \frac{1}{2}\{4\varphi(x_i)-\varphi(2x_i)\},$$

$$\int_0^1\int_0^1 K_2^I(t_1,t_2)x_i(t_1)x_j(t_2)dt_1dt_2 = \frac{1}{2}\{\varphi(x_i+x_j)-\varphi(x_i)-\varphi(x_j)\},\quad i,j=1,2 . \qquad (5)$$

We have a system of linear integral equations (5) for determining the kernels K_1^I, K_2^I. Obviously, this system has many solutions and, therefore, the interpolation polynomial $P_{2,2}^I(F;x)$ is not unique. Let us assume $\overline{P}_2(x)$ a fixed polynomial of the form

$$\overline{P}_2(x) = \overline{K}_0 + \int_0^1 \overline{K}_1(t)x(t)dt + \int_0^1\int_0^1 \overline{K}_2(t_1,t_2)x(t_1)x(t_2)dt_1dt_2 .$$

Then we use formulas (2) to determine the formula $P_{2,2}^I(\overline{P}_2;x)$ and obtain

$$K_0^I = \overline{K}_0,\quad \int_0^1 K_1^I(t)x_i(t)dt = \int_0^1 \overline{K}_1(t)x_i(t)dt,$$

$$\int_0^1\int_0^1 K_2^I(t_1,t_2)x_i(t_1)x_j(t_2)dt_1dt_2 = \int_0^1\int_0^1 \overline{K}_2(t_1,t_2)x_i(t_1)x_j(t_2)dt_1dt_2,\quad i,j=1,2 .$$

In general, these inequalities do not imply the inequalities $K_i^I = \overline{K}_i$, $i=0,1,2$, which does not provide the invariance of the interpolant $P_{2,2}^I(F;x)$ with respect to polynomials of integral form of the second degree.

Let $x_1, x_2, \ldots, x_m$ be the linearly independent elements of X. We construct the set of elements corresponding to these elements. $Z(m) = \{z_i\}_{i=0}^N$, $N = N(m)$. It is known from [7] that in the space X' adjoint to X there exists a system of linear functionals

$l_i(x)$, $i=1,2,\ldots,m$ biorthogonal (adjoint) to the system x_i, $i=1,2,\ldots,m$, so $l_i(x_j)=\delta_{ij}$, $i,j=1,2,\ldots,m$, δ_{ij} is the Kronecker symbol. We consider the set of operator polynomials of the n degree of the form

$$\pi_{m,n}=\left\{P_{m,n}:P_{m,n}(x)=\sum_{k=0}^{n}\sum_{i_1,\ldots,i_k=1}^{m}a_{i_1\ldots i_k}l_{i_1}(x)\ldots l_{i_k}(x)\right\}, \tag{6}$$

where $a_{i_1\ldots i_k}$ are symmetric elements from Y with respect to their indices.

Theorem 2. *We let the operator polynomial* $P^I_{m,n}\in\pi_{m,n}$, *where* $a^I_{i_1\ldots i_k}$ *defined by (2), when replacing* $L^I_k x^k$ *by* $\sum_{i_1,\ldots,i_k=1}^{m}a^I_{i_1\ldots i_k}l_{i_1}(x)\ldots l_{i_k}(x)$, $L^I_k(x_{i_1},\ldots,x_{i_k})$ *by* $a^I_{i_1\ldots i_k}$, $k=0,1,\ldots,n$, $1\le i_j\le m$. *Then this polynomial will be interpolating for* F *on the set of nodes* $Z(m)$ *with interpolation conditions (3). The interpolant* $P^I_{m,n}(F;x)$ *is unique and invariant with respect to polynomials of degree* n *on the set* $\pi_{m,n}$.

Proof. Since $a^I_{i_1\ldots i_k}$, defined by (2), are solutions of a linear system of equations equivalent to (3), the interpolativity of the polynomial $P^I_{m,n}(F;x)$ is obvious. Since this solution is unique, the corresponding interpolant $P^I_{m,n}(F;x)$ is also unique on the set $\pi_{m,n}$. Let $F(x)\equiv\overline{P}_{m,n}(x)\in\pi_{m,n}$. Then from formulas (2), taking into account the algebraic identities

$$(x_1+x_2+\ldots+x_n)^m-\Big[(x_1+x_2+\ldots+x_{n-1})^m+\ldots+$$
$$+(x_2+x_3+\ldots+x_n)^m\Big]+\Big[(x_1+x_2+\ldots+x_{n-2})^m+\ldots+$$
$$+(x_3+x_4+\ldots+x_n)^m\Big]+\ldots+(-1)^{m-1}\Big[x_1^m+x_2^m+\ldots+x_n^m\Big]=0, \tag{7}$$

$\forall m=1,2,\ldots,n-1$, that are applied to the operator polynomial $\overline{P}_{m,n}(x)$, we obtain

$$a^I_{i_1\ldots i_k}=\overline{a}_{i_1\ldots i_k},\quad k=0,1,\ldots,n,\quad 1\le i_j\le m, \tag{8}$$

where $\overline{a}_{i_1\ldots i_k}$ are the elements of Y, corresponding to the polynomial $\overline{P}_{m,n}(x)$. The equalities (8) mean the invariance of the interpolant $P^I_{m,n}(F;x)$ with respect to all polynomials of the set $\pi_{m,n}$: $P^I_{m,n}(\overline{P}_{m,n};x) \equiv \overline{P}_{m,n}(x)$. The theorem is proved.

Remark 1. If X is a pre-Hilbert space with scalar product $(\cdot,\cdot)$, then, according to the results of [8], $l_i(x)$ we can write it in the form $l_i(x)=\sum_{j=1}^{m}\alpha_{i,j}(x,x_j)$, $i=1,2,\ldots m$, where $\alpha_{i,j}$ are the elements of the inverse Gram matrix constructed by the system of linearly independent elements $x_1,x_2,\ldots,x_m$.

Remark 2. It follows from [8] that in a pre-Hilbert space X for the invariant solvability of an interpolation problem with conditions (3) (for the existence of an operator polynomial interpolant on the set of nodes $Z(m)$ at any values of $F(z_i)$, $i=0,1,\ldots,N$) it is necessary and sufficient to fulfill the condition $\hat{Z}=0$, where $\hat{Z}$ is a matrix whose rows are the coordinates of the orthonormal eigenvectors of the matrix $\Gamma=\left\|\sum_{k=0}^{n}(z_i,z_j)^k\right\|_{i,j=0}^{N}$, $0^0=1$. It is obvious that

$$\hat{Z}=0 \Leftrightarrow rank\,\Gamma = N+1.$$

But, since the interpolation problem on the set of nodes $Z(m)$ (with the interpolation conditions (3)) is invariantly solvable according to relations (2), then

$$rank\left\|\sum_{k=0}^{n}(z_i,z_j)^k\right\|_{i,j=0}^{N}=N+1.$$

11.2. Polynomial interpolation in $C[0,\infty)$

Let X,Y – Banach spaces, π_n be the set of continuous polynomials of the form (1). The following problem is to define a sequence of nodes and an interpolation polynomial of degree n for an operator $F:X\to Y$ on this sequence that would have the properties of uniqueness and invariance with respect to polynomials of the same degree over the whole set π_n. Let $\{x_i\}_{i=1}^{\infty}$ be the basis of the space X. We construct the set of nodes $Z(m)=\{z_i\}_{i=0}^{N}$, $N=N(m)$, where $z_0=0$, $z_i=x_i$, $i=1,2,\ldots,m$ are the first m elements of the basis. Based on [9], in a space X' conjugate to X there exists a system of linear continuous functionals $\{l_i(x)\}_{i=1}^{\infty}$ biorthogonal (conjugate) to the system of elements $\{x_i\}_{i=1}^{\infty}$, thus $l_i(x_j)=\delta_{ij}$, $i,j=1,2,\ldots$. We consider a sequence (by index m) of interpolation polynomials $P_{m,n}^{I}(F;x)$ on the set of nodes $Z(m)$ of the following form

$$P_{m,n}^{I}(F;x)=\sum_{k=0}^{n}\sum_{i_1,\ldots,i_k=1}^{m} a_{i_1\ldots i_k}^{I} l_{i_1}(x)\ldots l_{i_k}(x), \qquad (9)$$

where $a_{i_1\ldots i_k}^{I}$ are the elements of Y, which are symmetric with respect to their indices and are defined by formulas (4.2). Without reducing generality, we assume that $\|x_i\|=1$, $\|l_i\|=1$, $i=1,2,\ldots$. The following theorem holds.

Theorem 3. *Let*

$$\sum_{i_1,\ldots,i_k=1}^{\infty}\left\|a_{i_1\ldots i_k}^{I}\right\|\le M_k=const, \quad k=0,1,\ldots,n. \qquad (10)$$

Then the limiting polynomial $P_{\infty,n}^{I}(F;x)=\lim_{m\to\infty}P_{m,n}^{I}(F;x)\in\pi_n$, *exists* $\forall x\in X$, *is an interpolation polynomial for the operator* F *on a countable sequence of nodes* $Z(\infty)$, *unique on the set of* π_n *continuous polynomials of the form (1) and invariant with respect to all polynomials of degree* n *from* π_n.

Proof. *Existence of the limit.* From condition (10) of the theorem it follows that

$$\left\|\sum_{i_1,\ldots,i_k=1}^{\infty}a_{i_1\ldots i_k}^{I}l_{i_1}(x)\ldots l_{i_k}(x)\right\|\le\sum_{i_1,\ldots,i_k=1}^{\infty}\left\|a_{i_1\ldots i_k}^{I}\right\|\|x\|^k\le M_k\|x\|^k, \quad k=0,1,\ldots,n.$$

The latter means that the series in $P_{\infty,n}^{I}(F;x)$ converge, and the k operator degrees of

 ISBN 978-3-98924-038-4

this polynomial are continuous. Thus $P^I_{\infty,n}(F;x)$ exists, is continuous and belongs to π_n.

Interpolation. Since for every fixed m, $m \ge n$ the polynomial $P^I_{m,n}(F;x)$ is interpolative for F on a set of nodes $Z(m)$, $m = n, n+1, \dots$ and there is an embedding $Z(n) \subset Z(n+1) \subset \cdots$, the limiting polynomial will be interpolative on a countable sequence of nodes $Z(\infty)$.

The uniqueness of the polynomial $P^I_{\infty,n}(F;x)$ is obvious, since each element of the sequence $P^I_{m,n}(F;x)$ is determined by formulas (2) in the unique way.

Invariance. Let $F \equiv P_n = L_0 + L_1 x + \ldots + L_n x^n \in \pi_n$. Taking into account the algebraic identities (7), applied to the polynomial $P_n(x)$ and based on formulas (2), we obtain

$$P^I_{m,n}(P_n;x) = \sum_{k=0}^{n} \sum_{i_1,\dots,i_k=1}^{m} L_k\left(x_{i_1},\dots,x_{i_k}\right) l_{i_1}(x)\dots l_{i_k}(x).$$

Then

$$P^I_{\infty,n}(P_n;x) = \lim_{m\to\infty} P^I_{m,n}(P_n;x) = \lim_{m\to\infty} \sum_{k=0}^{n} \sum_{i_1,\dots,i_k=1}^{m} L_k\left(x_{i_1},\dots,x_{i_k}\right) l_{i_1}(x)\dots l_{i_k}(x) =$$

$$= \sum_{k=0}^{n} \lim_{m\to\infty} L_k\left(\sum_{i_1=1}^{m} l_{i_1}(x) x_{i_1}, \dots, \sum_{i_k=1}^{m} l_{i_k}(x) x_{i_k} \right) =$$

$$\sum_{k=0}^{n} L_k\left(\lim_{m\to\infty} \sum_{i_1=1}^{m} l_{i_1}(x) x_{i_1}, \dots, \lim_{m\to\infty} \sum_{i_k=1}^{m} l_{i_k}(x) x_{i_k} \right) =$$

$$= \sum_{k=0}^{n} L_k(x,\dots,x) = \sum_{k=0}^{n} L_k x^k = P_n(x) \in \pi_n,$$

which proves the invariance. Here, we use the representation of the element $x \in X$ in the form of $x = \sum_{i=1}^{\infty} l_i(x) x_i$, and the possibility of a limit transformation, since the k-linear operator forms of the polynomial $P_n(x)$ are continuous. The theorem is proved.

 ISBN 978-3-98924-038-4

Let us consider the set of polynomials of integral form defined on the space $C[0,\infty)$ of continuous functions with a domain of values on the real axis. We will construct a new type of interpolation polynomial with continuous nodes, which has the properties of uniqueness and invariance. The interpolation formula does not contain either the operation of differentiation of the interpolated functional or Stieltjes integrals, and does not require the "substitution rule", about which we will discuss below. In [10], the existence of an interpolation functional polynomial of the form is proved

$$P_n(x) = K_0 + \int_0^1 K_1(t)(x(t) - \varphi_0(t)) +$$

$$+ \int_0^1 \int_{t_1}^1 K_2(t_1,t_2)(x(t_1) - \varphi_0(t_1))(x(t_2) - \varphi_1(t_2)) dt_2 dt_1 + \ldots +$$

$$+ \int_0^1 \int_{t_1}^1 \ldots \int_{t_{n-1}}^1 K_n(t_1,t_2,\ldots,t_n)(x(t_1) - \varphi_0(t_1))(x(t_2) - \varphi_1(t_2)) \ldots \times$$

$$\times (x(t_n) - \varphi_{n-1}(t_n)) dt_n dt_{n-1} \ldots dt_1, \quad (11)$$

with kernels $K_i \in Q(\mathbf{\Omega}_i)$, where $\varphi_i(t) \in Q[0,1]$, $i = 0,\ldots,n-1$, $\mathbf{\Omega}_i = \{(\xi_1,\xi_2,\ldots,\xi_i):$ $0 \le \xi_1 \le \xi_2 \le \ldots \le \xi_i \le 1\}$, $Q(\Omega_i)$ is a set of piecewise continuous functions for each variable with a finite number of discontinuity points of the first kind. Moreover, the interpolation continuum node is the function

$$x^n(t,\boldsymbol{\xi}^n) = \varphi_0(t) + \sum_{i=1}^n H(t - \xi_i)(\varphi_i(t) - \varphi_{i-1}(t)), \quad \boldsymbol{\xi}^n \in \mathbf{\Omega}_n, \quad k = 1,2,\ldots,n$$

H is the Heaviside function, and the sufficient conditions for the existence of such an interpolant are as follows:

1. $$\prod_{i=1}^n [\varphi_i(\xi_i) - \varphi_{i-1}(\xi_i)]^{-1} \frac{\partial^n}{\partial \xi_1 \ldots \partial \xi_n} F(x^n(\cdot,\boldsymbol{\xi}^n)) \in Q(\mathbf{\Omega}_n), \quad (12)$$

2. "substitution rule"

$$\left\{ \frac{\partial}{\partial z_k} \left[\frac{\partial^{k-1}}{\partial z_1 \partial z_2 \ldots \partial z_{k-1}} F(x^{k-1}(\cdot,\boldsymbol{\xi}^{k-1}) + H(\cdot - z_k)(\varphi_k(\cdot) - \varphi_{k-1}(\cdot)) + \right.\right.$$

 ISBN 978-3-98924-038-4

$$+H(\cdot - z_{k+1})(\varphi_{k+1}(\cdot) - \varphi_k(\cdot)))]\}_{z_{k+1}=z_k} =$$

$$= \frac{\varphi_k(z_k) - \varphi_{k-1}(z_k)}{\varphi_{k+1}(z_k) - \varphi_{k-1}(z_k)} \frac{\partial}{\partial z_k}\left[\frac{\partial^{k-1}}{\partial z_1 \partial z_2 \ldots \partial z_{k-1}} F\left(x^{k-1}(\cdot, \boldsymbol{\xi}^{k-1}\right) +\right.$$

$$\left. + H(\cdot - z_k)(\varphi_{k+1}(\cdot) - \varphi_{k+1}(\cdot)))\right], \qquad k = 1,2,\ldots,n-1. \tag{13}$$

Here $\phi_i(t) \in Q[0,1]$, $\phi_i(t) - \phi_{i-1}(t) \neq \phi_j(t) - \phi_{j-1}(t)$, $i \neq j$. If $\phi_i(t) = \phi_2(t)$ then the "substitution rule" (13) is always satisfied.

Let us construct an interpolation functional polynomial of integral form with the above properties of uniqueness and invariance. We denote by π_n the set of functional polynomials of the degree n of the form

$$\pi_n = \left\{ P_n : P_n(x) = K_0 + \int_0^\infty K_1(t)x(t)dt + \int_0^\infty\int_0^\infty K_2(t_1,t_2)x(t_1)x(t_2)dt_1dt_2 + \ldots + \right.$$

$$+ \int_0^\infty \ldots \int_0^\infty K_n(t_1,t_2,\ldots,t_n)x(t_1)x(t_2)\ldots x(t_n)dt_1dt_2\ldots dt_n, \quad P_n : C[0,\infty) \to R^1, \tag{14}$$

where K_i are symmetric functions of their variables, $K_i \in L_1(\boldsymbol{\Omega}_i)$, $\Omega_i = [0,\infty) \times \ldots (i) \ldots [0,\infty)$, $x \in C[0,\infty)$. Let us introduce a system of linearly independent functions at different ξ_i:

$$x_i(t,\xi_i) = \phi(t)\sin t\xi_i, \quad i = 1,2,\ldots,n, \quad \phi(t) \in C[0,\infty). \tag{15}$$

We construct a set of continuum nodes $Z(n) = \{z_i\}_{i=0}^N$, $N = N(m)$, depending on the real parameters ξ_i.

The problem is to find a polynomial of degree n $P_n^I(F;x) \in \pi_n$ with kernels K_i^I, that satisfies the interpolation conditions

$$P_n^I(F; z_i(\cdot,\xi)) = F(z_i(\cdot,\xi)), \quad i = 0,1,2,\ldots,N, \ \forall \xi \in \Omega_n. \tag{16}$$

Note that instead of the conditions (4.16), we can set one interpolation condition on the continuum node $z^n(t,\xi) = \phi(t)\sum_{i=1}^n \sin(t\xi_i)$, so that, instead of (16), we require the fulfillment of the identity $P_n^I\left(F;z^n(\cdot,\xi)\right)=F\left(z^n(\cdot,\xi)\right)$, $\forall\xi\in\mathbf{\Omega}_n$, from which, at the corresponding values of ξ_i, as particular cases, the conditions (16) will follow. The choice of continual interpolation nodes in the form of a set $Z(n)$ based on (15) is convenient for further description.

Thus, we have the following problem. We need to define the kernels K_i^I, $i=0,1,\ldots,n$ so that the corresponding polynomial $P_n^I(F;x)$ is interpolative on a continuous set of nodes $z_i(t,\xi)$, $i-0,1,\ldots,N$, which depends on a continuous vector parameter $\xi\in\mathbf{\Omega}_n$ with interpolation conditions (16). According to (14) the operator degree p of the polynomial $P_n^I(F;x)$ will have the form

$$L_p^I x^p=\int_0^\infty\ldots\int_0^\infty K_p^I(t_1,t_2,\ldots,t_p)x(t_1)x(t_2)\ldots x(t_p)dt_1dt_2\ldots dt_p .$$

From the conditions (16), we find the values of p- linear operator forms $L_p^I\left(z_1(\cdot,\xi_1),z_2(\cdot,\xi_2),\ldots,z_p(\cdot,\xi_p)\right)$ as the right-hand sides of inequalities (2). On the other hand

$$L_p^I\left(z_1(\cdot,\xi_1),z_2(\cdot,\xi_2),\ldots,z_p(\cdot,\xi_p)\right)=$$

$$=\int_0^\infty\ldots\int_0^\infty K_p^I(t_1,t_2,\ldots,t_p)z_1(t_1,\xi_1)z_2(t_2,\xi_2)\ldots z_p(t_p,\xi_p)dt_1dt_2\ldots dt_p=$$

$$=\int_0^\infty\ldots\int_0^\infty K_p^I(t_1,t_2,\ldots,t_p)\prod_{i=1}^p\phi(t_i)\sin(t_i\xi_i)dt_i,\quad p=1,2,\ldots,n. \tag{17}$$

Applying the inverse sine transformation, we obtain from formula (17)

$$K_p^I(\xi_1,\xi_2,\ldots,\xi_p)=\left(\frac{2}{\pi}\right)^p\left[\prod_{i=1}^p\phi(\xi_i)\right]^{-1}\times$$

$$\times\int_0^\infty\ldots\int_0^\infty L_p^I\left(z_1(\cdot,t_1),z_2(\cdot,t_2),\ldots,z_p(\cdot,t_p)\right)\prod_{i=1}^p\sin(t_i\xi_i)dt_i,\quad p=1,2,\ldots,n. \tag{18}$$

Thus, if the conditions (16) are met, then the kernels K_p^I of the polynomial P_n^I

are determined by formulas (18). On the contrary, if the kernels are calculated by formulas (18), then the sine transformation determines the values of the p-linear operator forms

$$L_p^I\left(z_{i_1}\left(\cdot,\xi_{i_1}\right), z_{i_2}\left(\cdot,\xi_{i_2}\right),\dots,z_{i_p}\left(\cdot,\xi_{i_p}\right)\right),\quad p=1,2,\dots,n,\quad 1\le i_j\le n,$$

which, are the solution of a system of linear equations equivalent to the interpolation conditions (16).

Theorem 4. *Let the functional* $F(x(\cdot))$ *satisfy the conditions*

$$L_p^I\left(z_1(\cdot,t_1),\dots,z_p(\cdot,t_p)\right)\in L_1(\Omega_p),\quad p=1,2,\dots,n.$$

Then, in order for the operator polynomial $P_n^I(F;x)$ *to be interpolative on a continuous set of nodes* $Z(n)=\{z_i\}_{i=0}^N$, $N=N(n)$ *it is necessary and sufficient that its kernels are defined by formulas (18).*

We also note that the interpolant $P_n^I(F;x)$ on a continuous set of nodes is unique. This fact is obvious as a consequence of the result of the previous theorem. Furthermore, the following statement holds

Theorem 5. *The interpolation polynomial* $P_n^I(F;x)$ *on a continuous set of nodes* $Z(n)$ *is invariant with respect to all polynomials of the form (14).*

Proof. Let the interpolated operator F, be a functional polynomial $P_n\in\pi_n$ of the form (14), so $F\equiv P_n$. Then, using the algebraic identities (7) applied to the polynomial P_n, and the conditions

$$P_n^I(F;z_i)=\varphi(z_i),\quad \varphi(x)=P_n(x)-P_n(0),\quad i=1,2,\dots,N$$

we obtain by formulas (2) the values

$$L_p^I\left(z_1(\cdot,\xi_1),z_2(\cdot,\xi_2),\dots,z_p(\cdot,\xi_p)\right)=$$

$$=\int_0^\infty\dots\int_0^\infty K_p^I(t_1,t_2,\dots,t_p)z_1(t_1,\xi_1)z_2(t_2,\xi_2)\dots z_p(t_p,\xi_p)dt_1dt_2\dots dt_p,\quad i=1,2,\dots,n,\quad (19)$$

where K_p are the kernels of the polynomial $P_n(x)$. From (19), by the inverse sine transformation, we obtain

$$K_p(\xi_1,\xi_2,\dots,\xi_p)=\left(\frac{2}{\pi}\right)^p\left[\prod_{i=1}^{p}\phi(\xi_i)\right]^{-1}\times$$

$$\times\int_0^\infty\dots\int_0^\infty L_p^I(z_1(\cdot,t_1),z_2(\cdot,t_2),\dots,z_p(\cdot,t_p))\prod_{i=1}^{p}\sin(t_i\xi_i)dt_i=K_p(\xi_1,\xi_2,\dots,\xi_p),$$

$$p=1,2,\dots,n.$$

Thus, the polynomials P_n^I and P_n coincide, so $P_n^I(P_n;x)=P_n$. The theorem is proved.

Remarks 3. Instead of the sine transformation, any other integral transforms for which the inverse formulas are known (cosine, Hankel, Kantorovich-Lebedev, and others) could be used. Such an approach to the construction of interpolation functional polynomials was used by Yanovich L.A. and his students [8], but their constructions contained Stieltjes integrals by the operator of the scalar argument and are not interpolative at the continuum nodes.

11.3. Existence of a unique, invariant interpolation operator polynomial in Hilbert space

Let H be a separable Hilbert space with an orthonormalized basis $\{e_i\}_{i=\overline{1,\infty}}$, $(e_i,e_j)=\delta_{ij},\quad i,j=1,2,\dots$, where $(\cdot,\cdot)$ is a scalar product in H, is the Kronecker symbol. The interpolating operators act from H to the Banach space Y. The norm generated by the scalar product in H, is denoted by $\|\cdot\|$, and the norm in Y is denoted by $\|\cdot\|_Y$.

Let us fix any $n+1$ elements of H. Let them be $x_i,\quad i=\overline{0,n}$, and $x_i\neq x_j$, $i\neq j$. We introduce a countable set of points

$$x_{(\xi_1,\xi_2,\ldots,\xi_n)} = x_0 + \sum_{s=1}^{n}\sum_{p=\xi}^{\infty}\left(x_s - x_{s-1}, e_p\right)e_p, \qquad (20)$$

$$\boldsymbol{\xi}^n = (\xi_i)_{i=\overline{1,n}} \in \left\{\boldsymbol{\eta}^n = (\eta_i)_{i=\overline{1,n}} \mid 1 \le \eta_1 \le \eta_2 \le \ldots \le \eta_n \le \infty, \quad \eta_i \in \mathbb{N}, i = \overline{1,n}\right\} = \boldsymbol{\Omega}_n,$$

which we will later use as interpolation nodes. Here $\mathbb{N}$ is a set of positive integers.

Let us introduce a set of continuous operator polynomials of degree n

$$\Pi_n = \left\{P_n : H \to Y \mid P_n(x) = b + \sum_{i_1=1}^{\infty} b_{(i_1)}\left(x, e_{i_1}\right) + \ldots +\right.$$

$$+\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty} b_{(i_1,i_2,\ldots,i_n)}\left(x, e_{i_1}\right)\left(x, e_{i_2}\right)\ldots\left(x, e_{i_n}\right), \quad b_{(i_1,i_2,\ldots,i_k)} \in Y, \quad \forall i_1,\ldots,i_k \in \mathbb{N},$$

$$\left. k = \overline{1,n}, b \in Y, \quad \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_k=1}^{\infty}\left\|b_{(i_1,i_2,\ldots,i_k)}\right\|^2 < \infty, k = \overline{1,n}\right\} \qquad (21)$$

and formulate the following interpolation problem.

Problem I. For the operator $F(x)$, given its values on the set of nodes (20), find an operator polynomial $P_n^I(x) \in \Pi_n$, that satisfies the interpolation conditions

$$P_n^I\left(x_{(\xi_1,\xi_2,\ldots,\xi_n)}\right) = F\left(x_{(\xi_1,\xi_2,\ldots,\xi_n)}\right), \quad \boldsymbol{\xi}^n \in \boldsymbol{\Omega}_n. \qquad (22)$$

The following notations we will use:

$$\Delta_{i_1 i_2 \ldots i_k} F\left(x_{(i_1,i_2,\ldots,i_k)}\right) = -\nabla_{i_1 i_2 \ldots i_k} F\left(x_{(i_1,i_2,\ldots,i_k)}\right) =$$

$$= \Delta_{i_1 i_2 \ldots i_{k-1}} F\left(x_{(i_1,i_2,\ldots,i_{k-1},i_k+1)}\right) - \Delta_{i_1 i_2 \ldots i_{k-1}} F\left(x_{(i_1,i_2,\ldots,i_{k-1},i_k)}\right),$$

$$k = 1,2,\ldots,n+1, \quad \Delta_{i_1} F(x(i_1)) = F(x(i_1+1)) - F(x(i_1)) \qquad (23)$$

(recurrent definition of mixed differences forward of any order).

Theorem 6. *Let the continuous operator* $F(x)$ *satisfy the conditions*

$$\sum_{i_1=1}^{\infty}\sum_{i_2=i_1+1}^{\infty}\cdots\sum_{i_k=i_{k-1}+1}^{\infty}\frac{\left\|\Delta_{i_1 i_2 \ldots i_k} F\left(x_{(i_1,i_2,\ldots,i_k)}\right)\right\|_Y^2}{\left(x_1 - x_0, e_{i_1}\right)^2 \ldots \left(x_k - x_{k-1}, e_{i_k}\right)^2} < \infty, \qquad (24)$$

$$k = \overline{1,n+1}$$

where $i_0 = 0$, $x_i \in H$, $i = \overline{0, n+1}$, *a set of arbitrary elements. Then, in order for the operator* $F(x)$ *to have a representation*

$$F(x) = F(x_0) - \sum_{i_1=1}^{\infty} \frac{(x - x_0, e_{i_1})}{(x_1 - x_0, e_{i_1})} \Delta_{i_1} F(x(i_1)) + \ldots +$$

$$+ (-1)^n \sum_{i_1=1}^{\infty} \sum_{i_2=i_1+1}^{\infty} \cdots \sum_{i_n=i_{n-1}+1}^{\infty} \frac{(x - x_0, e_{i_1})}{(x_1 - x_0, e_{i_1})} \cdot \frac{(x - x_1, e_{i_2})}{(x_2 - x_1, e_{i_2})} \cdots \frac{(x - x_{n-1}, e_{i_n})}{(x_n - x_{n-1}, e_{i_n})} \cdot$$

$$\cdot \Delta_{i_1 i_2 \ldots i_n} F(x(i_1, i_2, \ldots i_n)) + R_n(x) = P_n^I(x) + R_n(x), \qquad (25)$$

where

$$R_n(x) = (-1)^{n+1} \sum_{i_1=1}^{\infty} \sum_{i_2=i_1+1}^{\infty} \cdots \sum_{i_n=i_{n-1}+1}^{\infty} \frac{(x - x_0, e_{i_1})}{(x_1 - x_0, e_{i_1})} \cdots \frac{(x - x_{n-1}, e_{i_n})}{(x_n - x_{n-1}, e_{i_n})} \cdot \frac{(x - x_n, e_{i_{n+1}})}{(x_{n+1} - x_n, e_{i_{n+1}})} \cdot$$

$$\cdot \Delta_{i_1 i_2 \ldots i_{n+1}} F(x(i_1, i_2, \ldots, i_{n+1})), \quad x_{n+1} = x, \qquad (26)$$

is necessary and sufficient for the discrete analog of the substitution rule to hold

$$\Delta_{i_1 i_2 \ldots i_k} F\left(x_{(i_1, i_2, \ldots, i_k, i_k)}\right) = \frac{(x_{k+1} - x_{k-1}, e_{i_k})}{(x_k - x_{k-1}, e_{i_k})} \cdot \left[\Delta_{i_1 i_2 \ldots i_k} F\left(x_{(i_1, i_2, \ldots, i_k, i_{k+1})}\right)\right]_{i_{k+1} = i_k + 1}$$

$$k = \overline{1, n}. \qquad (27)$$

Proof. *Sufficiency*. Let (27) be satisfied. Then, considering (24), the following transformations will be valid

$$R_n(x) = (-1)^{n+1} \sum_{i_1=1}^{\infty} \sum_{i_2=i_1+1}^{\infty} \cdots \sum_{i_n=i_{n-1}+1}^{\infty} \frac{(x - x_0, e_{i_1})}{(x_1 - x_0, e_{i_1})} \cdots \frac{(x - x_{n-1}, e_{i_n})}{(x_n - x_{n-1}, e_{i_n})} \cdot$$

$$\cdot \sum_{i_{n+1}=i_n+1}^{\infty} \Delta_{i_1 i_2 \ldots i_{n+1}} F\left(x_{(i_1, i_2, \ldots, i_{n+1})}\right) = -(-1)^n \sum_{i_1=1}^{\infty} \sum_{i_2=i_1+1}^{\infty} \cdots \sum_{i_n=i_{n-1}+1}^{\infty} \frac{(x - x_0, e_{i_1})}{(x_1 - x_0, e_{i_1})} \cdots \frac{(x - x_{n-1}, e_{i_n})}{(x_n - x_{n-1}, e_{i_n})} \cdot$$

$$\cdot \left\{\left[\Delta_{i_1 i_2 \ldots i_n} F\left(x_{(i_1, i_2, \ldots, i_{n+1})}\right)\right]_{i_{n+1} = i_n + 1} - \Delta_{i_1 i_2 \ldots i_n} F\left(x_{(i_1, i_2, \ldots, i_n)}\right)\right\} = R_{n-1}(x) -$$

$$-(-1)^n \sum_{i_1=1}^{\infty} \sum_{i_2=i_1+1}^{\infty} \cdots \sum_{i_n=i_{n-1}+1}^{\infty} \frac{(x - x_0, e_{i_1})}{(x_1 - x_0, e_{i_1})} \cdots \frac{(x - x_{n-1}, e_{i_n})}{(x_n - x_{n-1}, e_{i_n})} \cdot \Delta_{i_1 i_2 \ldots i_n} F\left(x_{(i_1, i_2, \ldots, i_n)}\right).$$

Substituting the last expression in (25), we obtain

$$P_n^I(x) + R_n(x) = P_{n-1}^I(x) + R_{n-1}(x),$$

 ISBN 978-3-98924-038-4

where on the right-hand side, instead of x_n we substitute x.

Continuing on, we come to a chain of equalities

$$P_n^I(x)+R_n(x)=P_{n-1}^I(x)+R_{n-1}(x)=\ldots=P_0^I(x)+R_0(x)=$$

$$=F(x_0)-\sum_{i_1=1}^{\infty}\frac{(x-x_0,e_{i_1})}{(x_1-x_0,e_{i_1})}\Delta_{i_1}F(x_{(i_1)})=F(x_0)-\sum_{i_1=1}^{\infty}\Delta_{i_1}F(x_{(i_1)})=$$

$$=F(x_0)+F(x_{(1)})-F(x_\infty)=F(x).$$

Here, in the last step, we used that $x_{(1)}=x_0+\sum_{p=1}^{\infty}(x_1-x_0,e_p)e_p=x_1=x, \quad x_\infty=x_0$ and the continuity of the operator $F(x)$. The sufficiency is proved.

Necessity. Let representation (25) be true. The conditions (24) guarantee this statement. We substitute $x=x_{(\xi_1,\xi_2,\ldots,\xi_{n-1},\xi_{n-1})}$ into (25) and take from both parts the difference $\Delta_{\xi_1\xi_2\ldots\xi_{n-1}}$ of the order $(n-1)$. Then we obtain

$$\Delta_{\xi_1\xi_2\ldots\xi_{n-1}}F(x_{(\xi_1,\xi_2,\ldots,\xi_{n-1},\xi_{n-1})})=\frac{(x_n-x_{n-2},e_{i_{n-1}})}{(x_{n-1}-x_{n-2},e_{i_{n-1}})}\cdot\left[\Delta_{\xi_1\xi_2\ldots\xi_{n-1}}F(x_{(\xi_1,\xi_2,\ldots,\xi_{n-1},\xi_n)})\right]_{\xi_n=\xi_n+1},$$

which means we derive a discrete analog of the substitution rule. The theorem is proved.

In the following theorem we consider the conditions for the existence of an interpolation operator polynomial of degree n on a countable number of interpolation nodes $x_{(i_1,\ldots,i_n)}$, $1\le i_1\le i_2\le\ldots\le i_n\le\ \le\infty$ of the form

$$P_n(x)=a_0+\sum_{i_1=1}^{\infty}a_{i_1}(x-x_0,e_{i_1})+\sum_{i_1=1}^{\infty}\sum_{i_2=i_1+1}^{\infty}a_{i_1i_2}(x-x_0,e_{i_1})\cdot(x-x_1,e_{i_2})+\ldots+$$

$$+\sum_{i_1=1}^{\infty}\sum_{i_2=i_1+1}^{\infty}\ldots\sum_{i_n=i_{n-1}+1}^{\infty}a_{i_1,i_2,\ldots,i_n}(x-x_0,e_{i_1})(x-x_1,e_{i_2})\ldots(x-x_{n-1},e_{i_n}) \tag{28}$$

for the operator $F:H\to Y$.

Theorem 7. *Let the operator $F(x)$ satisfy the conditions (24). Then, in order for there to be a unique operator interpolation polynomial of the form (28) on a countable*

 ISBN 978-3-98924-038-4

number of interpolation nodes $x_{(\xi_1,\xi_2,\ldots,\xi_n)}$, $\xi^n \in \Omega_n$ *for the operator* $F(x)$, *it is necessary and sufficient that a discrete analog of the substitution rule (25) holds.*

Proof. Let there be a discrete analog of the substitution rule (25). Then the operator polynomial $P_n^I(x)$ (28) with elements defined by the formulas

$$a_{i_1 i_2 \ldots i_k} = \frac{(-1)^k \Delta_{i_1 i_2 \ldots i_k} F(x(i_1,i_2,\ldots,i_k))}{(x_1 - x_0, e_{i_1})(x_2 - x_1, e_{i_2})\ldots(x_k - x_{k-1}, e_{i_k})}, \tag{29}$$

$$k = 1,2,\ldots,n, \quad a_0 = F(x_0)$$

will be the only interpolation operator polynomial of the form (28) of the degree n for the operator $F(x)$ at a countable number of interpolation nodes $x_{(i_1,i_2,\ldots,i_k)}$, $1 \le i_1 \le i_2 \le \ldots \le i_n \le \infty$. It follows from Theorem 6 that the representation (26) holds. Let us find the value of the residual term $R_n(x)$ of the element $x_{(\xi_1,\xi_2,\ldots,\xi_n)}$, where $\xi_1,\xi_2,\ldots,\xi_n$ are arbitrary positive integers satisfying the inequalities $1 \le \xi_1 \le$ $\le \xi_2 \le \ldots \le \xi_n \le \infty$.

Since

$$x_{(i_1,i_2,\ldots,i_n,i_{n+1})}\Big|_{x=x_{(\xi_1,\xi_2,\ldots,\xi_n)}} = x_0 + \sum_{s=1}^{n}\sum_{p=i_s}^{\infty}(x_s - x_{s-1}, e_p)e_p +$$
$$+ \sum_{p=i_{n+1}}^{\infty}(x(\xi_1,\xi_2,\ldots,\xi_n) - x_n, e_p)e_p \tag{30}$$

and

$$R_n\left(x_{(\xi_1,\xi_2,\ldots,\xi_n)}\right) = \sum_{i_1=\xi_1}^{\infty}\sum_{i_2=\xi_2}^{\infty}\cdots\sum_{i_n=\xi_n}^{\infty}\sum_{i_{n+1}=i_n+1}^{\infty}\prod_{s=1}^{n+1}\frac{\left(x_{(\xi_1,\xi_2,\ldots,\xi_n)} - x_{s-1}, e_{i_{s-1}}\right)}{\left(x_s - x_{s-1}, e_{i_{s-1}}\right)}\cdot$$
$$\cdot \Delta_{i_1,i_2,\ldots,i_{n+1}} F\left(x_{(i_1,i_2,\ldots,i_{n+1})}\right)\Big|_{x=x_{(\xi_1,\xi_2,\ldots,\xi_n)}},$$

then $\xi_s \le i_s$, $s = \overline{1,n}$, $\xi_n < i_{n+1}$ and from (30) we obtain the expression

$$x_{(i_1,i_2,\ldots,i_n,i_{n+1})}\Big|_{x=x(\xi_1,\xi_2,\ldots,\xi_n)} = x_0 + \sum_{s=1}^{n}\sum_{p=i_s}^{\infty}(x_s - x_{s-1}, e_p)e_p,$$

which is independent of i_{n+1}. Therefore $R_n\left(x_{(\xi_1,\xi_2,\ldots,\xi_n)}\right) = 0$, $1 \le \xi_1 \le \xi_2 \le \ldots \le \xi_n \le \infty$ and

ISBN 978-3-98924-038-4

$F\left(x_{(\xi_1,\xi_2,\ldots,\xi_n)}\right)=P_n^I\left(x_{(\xi_1,\xi_2,\ldots,\xi_n)}\right)$. The uniqueness is obvious.

The converse statement. Suppose that there exists a unique interpolation operator polynomial of degree n for $F(x)$ on a countable number of interpolation nodes $x_{(\xi_1,\xi_2,\ldots,\xi_n)}$, $1\le\xi_1\le\xi_2\le\ldots\le\xi_n\le\infty$, $\xi_i\in\mathbb{N}$, $i=\overline{1,n}$ of the form (28).

Then its elements are obviously determined by formulas (29). Let us show that the discrete analog of the substitution rule (27) will then hold. From the interpolation conditions, we obtain

$$\Delta_{\xi_1,\xi_2,\ldots,\xi_{n-1}}F\left(x_{(\xi_1,\xi_2,\ldots,\xi_{n-1},\xi_{n-1})}\right)\equiv\Delta_{\xi_1,\xi_2,\ldots,\xi_{n-1}}P_n^I\left(x_{(\xi_1,\xi_2,\ldots,\xi_{n-1},\xi_{n-1})}\right)=$$

$$=\Delta_{\xi_1,\xi_2,\ldots,\xi_{n-1}}\left\{P_{n-1}^I\left(x_{(\xi_1,\xi_2,\ldots,\xi_{n-1},\xi_{n-1})}\right)+\right.$$

$$\left.+\sum_{i_1=\xi_1}^{\xi_2-1}\sum_{i_2=\xi_2}^{\xi_3-1}\cdots\sum_{i_{n-2}=\xi_{n-2}}^{\xi_{n-1}}\sum_{i_{n-1}=\xi_{n-1}}^{\infty}\sum_{i_n=i_{n-1}+1}^{\infty}\Delta_{i_1i_2\ldots i_n}F\left(x_{(i_1,i_2,\ldots,i_n)}\right)\frac{\left(x_n-x_{n-2},e_{i_{n-1}}\right)}{\left(x_{n-1}-x_{n-2},e_{i_{n-1}}\right)}\right\}=$$

$$=\Delta_{\xi_1\xi_2\ldots\xi_{n-1}}\left[\sum_{i_1=\xi_1}^{\xi_2-1}\sum_{i_2=\xi_2}^{\xi_3-1}\cdots\sum_{i_{n-2}=\xi_{n-2}}^{\xi_{n-1}-1}\sum_{i_{n-1}=\xi_{n-1}}^{\infty}\Delta_{i_1i_2\ldots i_{n-1}}F\left(x_{(i_1,i_2,\ldots,i_{n-1})}\right)\cdot\frac{\left(x_n-x_{n-2},e_{i_{n-1}}\right)}{\left(x_{n-1}-x_{n-2},e_{i_{n-1}}\right)}\right]+$$

$$+\sum_{i_n=\xi_{n-1}+1}^{\infty}\Delta_{\xi_1,\xi_2,\ldots\xi_{n-1},i_n}F\left(x_{(\xi_1,\xi_2,\ldots\xi_{n-1},i_n)}\right)\cdot\frac{\left(x_n-x_{n-2},e_{i_{n-1}}\right)}{\left(x_{n-1}-x_{n-2},e_{i_{n-1}}\right)}=$$

$$=\Delta_{\xi_1,\xi_2,\ldots,\xi_{n-1}}F\left(x_{(\xi_1,\xi_2,\ldots,\xi_{n-1})}\right)\frac{\left(x_n-x_{n-2},e_{\xi_{n-1}}\right)}{\left(x_{n-1}-x_{n-2},e_{\xi_{n-1}}\right)}+$$

$$+\left\{\left[\Delta_{\xi_1,\xi_2,\ldots,\xi_{n-1}}F\left(x_{(\xi_1,\xi_2,\ldots,\xi_{n-1},i_n)}\right)\right]\Big|_{i_n=\xi_{n-1}+1}-\Delta_{\xi_1,\xi_2,\ldots,\xi_{n-1}}F\left(x_{(\xi_1,\xi_2,\ldots,\xi_{n-1})}\right)\right\}\frac{\left(x_n-x_{n-2},e_{\xi_{n-1}}\right)}{\left(x_{n-1}-x_{n-2},e_{\xi_{n-1}}\right)}.$$

Hence, we derive

$$\Delta_{\xi_1\xi_2\ldots\xi_{n-1}}F\left(x_{(\xi_1,\xi_2,\ldots,\xi_{n-1},\xi_{n-1})}\right)=\frac{\left(x_n-x_{n-2},e_{i_{n-1}}\right)}{\left(x_{n-1}-x_{n-2},e_{i_{n-1}}\right)}\cdot\left[\Delta_{\xi_1\xi_2\ldots\xi_{n-1}}F\left(x_{(\xi_1,\xi_2,\ldots,\xi_{n-1},i_n)}\right)\right]\Big|_{i_n=\xi_n+1},$$

which shows that the substitution rule holds. The theorem is completely proved.

Remark 4. The interpolation operator polynomial (28), (29) is a Newton-type interpolation polynomial in the sense that during the transition from $P_n^I(x)$ to $P_{n+1}^I(x)$ it is only necessary to add a new term to the polynomial $P_n^I(x)$.

 ISBN 978-3-98924-038-4

Remark 5. The following corollary follows from Theorem 6. We assume that the discrete analog of the substitution rule (25) is satisfied for the operator $F(x)$ with properties (24). Then the interpolation operator polynomial (28), (29) on a countable number of interpolation nodes (20) has the property of saving a polynomial of degree not exceeding n. The proof follows from the representation (26).

11.4. Interpolation operator polynomial of the third degree on a countable set of nodes

In the previous subsection, we investigated the polynomial interpolation of operators acting from a separable Hilbert space H into a Banach space Y. For the obtained polynomial to be interpolative for the operator $F(x)$ it is necessary and sufficient that the discrete analog of the substitution rule is fulfilled. However, the substitution rule imposes significant restrictions on the operator $F(x)$.

Let us formulate Problem 2 of this subsection. It is necessary to find an operator polynomial $P_3^I(x)$ for the operator $F(x)$ that satisfies the interpolation conditions

$$P_3^I\left(x_{(k_1,k_2,k_3)}\right)=F\left(x_{(k_1,k_2,k_3)}\right),\quad \forall k_i\in\mathbb{N},\quad 1\le k_1\le k_2\le k_3\le\infty,$$

where ℕ – is a set of positive integers,

$$x_{(k_1,k_2,k_3)}=x_0+\sum_{p=k_1}^{\infty}\left(x_1-x_0,e_p\right)e_p+\sum_{p=k_2}^{\infty}\left(x_2-x_1,e_p\right)e_p+\sum_{p=k_3}^{\infty}\left(x_3-x_2,e_p\right)e_p \tag{31}$$

is a countable interpolation node; $\{e_i\}_{i=\overline{1,\infty}}$, $\left(e_i,e_j\right)=\delta_{ij},\ i,j=1,2,\ldots$ is an orthonormalized basis; $(\cdot,\cdot)$ – a scalar product in H; δ_{ij} – Kronecker's symbol; $x_i, i=\overline{0,3}$ – arbitrary fixed elements of H, and $x_i\ne x_j$, $i\ne j$.

First, let us demonstrate the specificity of proving the existence and uniqueness theorem of the first degree interpolation operator polynomial on a countable set of interpolation nodes

$$x_{(\xi_1)} = x_0 + \sum_{p=\xi_1}^{\infty}(x_1 - x_0, e_p)e_p, \quad \xi_1 \in \mathbb{N},\ 1 \le \xi_1 \le \infty,\ (e_i, e_j) = \delta_{ij}. \tag{32}$$

Assume that the operator $F: H \to Y$ is continuous and that

$$\sum_{i_1=1}^{\infty}\frac{\left\|\nabla_{i_1}F\left(x_{(i_1)}\right)\right\|_Y^2}{\left(x_1 - x_0, e_{i_1}\right)^2} < \infty \tag{33}$$

and for all $(x_1 - x_0, e_{i_1}) \ne 0$, $i_1 = 1,2,\dots$. We will use the notation from the previous subsection.

Then the operator polynomial of the first degree

$$P_1^I(x) = F(x_0) + \sum_{i_1=1}^{\infty}\nabla_{i_1}F\left(x_{(i_1)}\right)\frac{\left(x - x_0, e_{i_1}\right)}{\left(x_1 - x_0, e_{i_1}\right)} \tag{34}$$

will be interpolating for the operator $F(x)$ on the countable set of nodes $x_{(\xi_1)}$.

First, we verify that the operator (34) is valid for any x from H. We obtain

$$\left\|P_1^I(x)\right\|_Y \le \left\|F(x_0)\right\|_Y + \left\{\sum_{i_1=1}^{\infty}\frac{\left\|\nabla_{i_1}F\left(x_{(i_1)}\right)\right\|_Y^2}{\left(x_1 - x_0, e_{i_1}\right)^2}\right\}^{1/2} \cdot$$

$$\cdot\left\{\sum_{i_1=1}^{\infty}\left(x - x_0, e_{i_1}\right)^2\right\}^{1/2} = \left\|F(x_0)\right\|_Y + \left\{\sum_{i_1=1}^{\infty}\frac{\left\|\nabla_{i_1}F\left(x_{(i_1)}\right)\right\|_Y^2}{\left(x_1 - x_0, e_{i_1}\right)^2}\right\}^{1/2}\left\|x - x_0\right\|, \quad \forall x \in H,$$

which, with condition (33), proves the above, and in addition, the continuity of the polynomial $P_1^I(x)$.

More

$$P_1^I\left(x_{(\xi_1)}\right) = F(x_0) + \sum_{i_1=\xi_1}^{\infty}\Delta_{i_1}F\left(x_{(i_1)}\right) = F(x_0) + F\left(x_{(\xi_1)}\right) - F\left(x_{(\infty)}\right) =$$

$$= F(x_0) + F\left(x_{(\xi_1)}\right) - F(x_0) = F\left(x_{(\xi_1)}\right), \quad \xi_1 = 1,2,\dots.$$

Here we have used the fact that the operator $F(x)$ is continuous and the series

$x_1 - x_0 = \sum_{p=1}^{\infty}(x_1 - x_0, e_p)e_p$ is convergent in the norm of the space H, so $\lim_{i_1\to\infty} x_{(i_1)} = x_0$.

Thus, we have proved that expression (34) is a continuous interpolation polynomial of first degree on a countable set of nodes $x_{(\xi_1)}$ for the operator $F(x)$.

Theorem 8. *We assume that condition (33) is satisfied. In order for an operator polynomial* $P_1^I(x) = F(x_0) + \sum_{i_1=1}^{\infty} a_{i_1}(x - x_0, e_{i_1})$ *be a continuous interpolation polynomial for a functional* F *on a countable set of nodes (32), it is necessary and sufficient that the formula* $a_{i_1} = \dfrac{F(x_{(i_1)}) - F(x_{(i_1+1)})}{(x_1 - x_0, e_{i_1})}$, $i_1 = 1,2,\dots$ *holds.*

Polynomial of the second degree. Let us consider a countable interpolation node

$$x_{(k,m)} = x_0 + \sum_{p=k}^{\infty}(x_1 - x_0, e_p)e_p + \sum_{p=m}^{\infty}(x_2 - x_1, e_p)e_p, \quad 1 \le k \le m \le \infty. \tag{35}$$

Then the operator polynomial of the second degree

$$P_2^I(x) = F(x_0) + \sum_{i_1=1}^{\infty} a_{i_1}(x - x_0, e_{i_1}) + \sum_{i_1=1}^{\infty}\sum_{i_2=i_1}^{\infty} a_{i_1,i_2}(x - x_0, e_{i_1})(x - x_1, e_{i_2}) \tag{36}$$

will be interpolative at the counted node (35), if its components are determined by the formulas

$$a_{i_1} = \frac{F(x_{(i_1)}) - F(x_{(i_1+1)})}{(x_1 - x_0, e_{i_1})},$$

$$a_{i_1,i_2} = \frac{F(x_{(i_1,i_2)}) - F(x_{(i_1+1,i_2)}) - F(x_{(i_1,i_2+1)}) + F(x_{(i_1+1,i_2+1)})}{(x_1 - x_0, e_{i_1})(x_2 - x_1, e_{i_2})}, \quad 1 \le i_1 \le i_2 - 1,$$

$$a_{i_1,i_1} = \left\{F(x_{(i_1,i_1)}) - F(x_{(i_1+1,i_1+1)}) - \frac{(x_2 - x_0, e_{i_1})}{(x_1 - x_0, e_{i_1})}\left[F(x_{(i_1,i_1+1)}) - F(x_{(i_1+1,i_1+1)})\right]\right\} \times$$
$$\times (x_2 - x_0, e_{i_1})^{-1}(x_2 - x_1, e_{i_1})^{-1}, \quad 1 \le i_1. \tag{37}$$

We obtain

$$P_2^I\left(x_{(k,m)}\right)=F\left(x_0\right)+F\left(x_{(k)}\right)-F\left(x_0\right)+\sum_{i_1=m}^{\infty}\frac{\left(x_2-x_1,e_{i_1}\right)}{\left(x_1-x_0,e_{i_1}\right)}\left[F\left(x_{(i_1)}\right)-F\left(x_{(i_1+1)}\right)\right]+$$

$$+\sum_{i_1=k}^{m-1}\sum_{i_2=m}^{\infty}\left[F\left(x_{(i_1,i_2)}\right)-F\left(x_{(i_1+1,i_2)}\right)-F\left(x_{(i_1,i_2+1)}\right)+F\left(x_{(i_1+1,i_2+1)}\right)\right]+$$

$$+\sum_{i_1=m}^{\infty}\sum_{i_2=i_1+1}^{\infty}\frac{\left(x_2-x_0,e_{i_1}\right)}{\left(x_1-x_0,e_{i_1}\right)}\left[F\left(x_{(i_1,i_2)}\right)-F\left(x_{(i_1+1,i_2)}\right)-F\left(x_{(i_1,i_2+1)}\right)+F\left(x_{(i_1+1,i_2+1)}\right)\right]+$$

$$+\sum_{i_1=m}^{\infty}\left\{F\left(x_{(i_1,i_1)}\right)-F\left(x_{(i_1+1,i_1+1)}\right)-\frac{\left(x_2-x_0,e_{i_1}\right)}{\left(x_1-x_0,e_{i_1}\right)}\left[F\left(x_{(i_1,i_1+1)}\right)-F\left(x_{(i_1+1,i_1+1)}\right)\right]\right\}=$$

$$=F\left(x_{(k)}\right)+\sum_{i_1=m}^{\infty}\frac{\left(x_2-x_1,e_{i_1}\right)}{\left(x_1-x_0,e_{i_1}\right)}\left[F\left(x_{(i_1)}\right)-F\left(x_{(i_1+1)}\right)\right]+$$

$$+\sum_{i_1=k}^{m-1}\left[F\left(x_{(i_1,m)}\right)-F\left(x_{(i_1+1,m)}\right)-F\left(x_{(i_1)}\right)+F\left(x_{(i_1+1)}\right)\right]+$$

$$+\sum_{i_1=m}^{\infty}\frac{\left(x_2-x_0,e_{i_1}\right)}{\left(x_1-x_0,e_{i_1}\right)}\left[F\left(x_{(i_1,i_1+1)}\right)-F\left(x_{(i_1+1,i_1+1)}\right)-F\left(x_{(i_1)}\right)+F\left(x_{(i_1+1)}\right)\right]+$$

$$+\sum_{i_1=m}^{\infty}\left\{F\left(x_{(i_1,i_1)}\right)-F\left(x_{(i_1+1,i_1+1)}\right)-\frac{\left(x_2-x_0,e_{i_1}\right)}{\left(x_1-x_0,e_{i_1}\right)}\left[F\left(x_{(i_1,i_1+1)}\right)-F\left(x_{(i_1+1,i_1+1)}\right)\right]\right\}=$$

$$=F\left(x_{(k)}\right)-\sum_{i_1=m}^{\infty}\left[F\left(x_{(i_1)}\right)-F\left(x_{(i_1+1)}\right)\right]+F\left(x_{(k,m)}\right)-F\left(x_{(m,m)}\right)-$$

$$-F\left(x_{(k)}\right)+F\left(x_{(m)}\right)+F\left(x_{(m,m)}\right)-F\left(x_{(0)}\right)=F\left(x_{(k,m)}\right),$$

which is what we needed to prove.

The following theorem is valid.

Theorem 9. *In order for the operator polynomial (36) to be interpolative for the operator* $F:H\to Y$ *on the countable node (35), it is necessary and sufficient that its elements are determined by formulas (37).*

Theorem 10. *Let the operator* $F:H\to Y$ *be such that for fixed* x_0, x_1 *and arbitrary* x *from* H *the following holds*

$$\sum_{i_1=1}^{\infty}\frac{\left\|\nabla_{i_1}F\left(x_{(i_1)}\right)\right\|_Y^2}{\left(x_1-x_0,e_{i_1}\right)^2}<\infty, \qquad \sum_{i_1=1}^{\infty}\sum_{i_2=i_1+1}^{\infty}\frac{\left\|\nabla_{i_1i_2}F\left(x_{(i_1,i_2)}\right)\right\|_Y^2}{\left(x_1-x_0,e_{i_1}\right)^2\left(x_2-x_1,e_{i_2}\right)^2}<\infty,$$

$$\sum_{i_1=1}^{\infty}\left\|\frac{\nabla_{i_1}F\left(x_{(i_1,i_1)}\right)}{\left(x_2-x_0,e_{i_1}\right)}-\frac{\left[\nabla_{i_1}F\left(x_{(i_1,i_2)}\right)\right]_{i_2=i_1+1}}{\left(x_1-x_0,e_{i_1}\right)}\right\|_Y^2<\infty,$$

where $x_2=x$. *Then the correct representation for this operator is*

$$F(x)=F(x_0)+\sum_{i_1=1}^{\infty}\left[\nabla_{i_1}F\left(x_{(i_1)}\right)\right]\frac{\left(x-x_0,e_{i_1}\right)}{\left(x_1-x_0,e_{i_1}\right)}+$$

$$+\sum_{i_1=1}^{\infty}\sum_{i_2=i_1+1}^{\infty}\left[\nabla_{i_1,i_2}F\left(x_{(i_1,i_2)}\right)\right]\frac{\left(x-x_0,e_{i_1}\right)\left(x-x_1,e_{i_2}\right)}{\left(x_1-x_0,e_{i_1}\right)\left(x_2-x_1,e_{i_2}\right)}+$$

$$+\sum_{i_1=1}^{\infty}\left\{\nabla_{i_1}F\left(x_{(i_1,i_1)}\right)-\frac{\left(x_2-x_0,e_{i_1}\right)}{\left(x_1-x_0,e_{i_1}\right)}\left[\nabla_{i_1}F\left(x_{(i_1,i_2)}\right)\right]_{i_2=i_1+1}\right\}.$$

The proof is carried out by direct verification.

Let us proceed to the solution of the formulated problem 2. The interpolation polynomial of the third degree will be obtained in the form

$$P_3^I(x)=p_0^I(x)+p_1^I(x)+p_2^I(x)+p_3^I(x), \tag{38}$$

where $p_0^I(x)=F(x_0),\quad p_1^I(x)=\sum_{i_1=1}^{\infty}a_{1,0,0}\left(x-x_0,e_{i_1}\right)$,

$$p_2^I(x)=p_{2,0,0}(x)+p_{2,0,1}(x)=\sum_{i_1=1}^{\infty}\sum_{i_2=i_1+1}^{\infty}a_{2,0,0}\left(x-x_0,e_{i_1}\right)\left(x-x_1,e_{i_2}\right)+$$

$$+\sum_{i_1=1}^{\infty}a_{2,0,1}\left(x-x_0,e_{i_1}\right)\left(x-x_1,e_{i_2}\right),$$

$$p_3^I(x)=\sum_{k=0}^{2}p_{3,0,k}(x)+\sum_{k=1}^{2}p_{3,1,k}(x)=$$

$$\sum_{i_1=1}^{\infty}\sum_{i_2=i_1+1}^{\infty}\sum_{i_3=i_2+1}^{\infty}a_{3,0,0}\left(x-x_0,e_{i_1}\right)\left(x-x_1,e_{i_2}\right)\left(x-x_2,e_{i_3}\right)+$$

$$+\sum_{i_1=1}^{\infty}\sum_{i_2=i_1+1}^{\infty}a_{3,0,1}\left(x-x_0,e_{i_1}\right)\left(x-x_1,e_{i_2}\right)\left(x-x_2,e_{i_2}\right)+$$

$$\sum_{i_1=1}^{\infty}a_{3,0,2}\left(x-x_0,e_{i_1}\right)\left(x-x_1,e_{i_1}\right)\left(x-x_2,e_{i_1}\right)+$$

$$+\sum_{i_1=1}^{\infty}\sum_{i_2=i_1+1}^{\infty}a_{3,1,1}\left(x-x_0,e_{i_1}\right)\left(x-x_1,e_{i_2}\right)\left(x-x_2,e_{i_2}\right)+$$

$$\sum_{i_1=1}^{\infty}a_{3,1,2}\left(x-x_0,e_{i_1}\right)\left(x-x_1,e_{i_1}\right)\left(x-x_2,e_{i_1}\right).$$

The following theorem is valid.

Theorem 11. *In order for the polynomial (38) to be interpolative for the operator* $F:H\to Y$ *on the countable node (31), it is necessary and sufficient that its elements are defined by the formulas*

$$a_{1,0,0}=\frac{\nabla_{i_1}F\left(x_{(i_1)}\right)}{\left(x_1-x_0,e_{i_1}\right)},\qquad a_{2,0,0}=\frac{\nabla_{i_1,i_2}F\left(x_{(i_1,i_2)}\right)}{\left(x_1-x_0,e_{i_1}\right)\left(x_2-x_1,e_{i_2}\right)},$$

$$a_{2,0,1}=\left[\frac{\left(x_1-x_0,e_{i_1}\right)}{\left(x_2-x_0,e_{i_1}\right)}\nabla_{i_1}F\left(x_{(i_1,i_1)}\right)-\nabla_{i_1}F\left(x_{(i_1,i_2)}\right)\Big|_{i_2=i_1+1}\right]\left(x_1-x_0,e_{i_1}\right)^{-1}\left(x_2-x_1,e_{i_2}\right)^{-1},$$

$$a_{3,0,0}=\frac{\nabla_{i_1,i_2,i_3}F\left(x_{(i_1,i_2,i_3)}\right)}{\left(x_1-x_0,e_{i_1}\right)\left(x_2-x_1,e_{i_2}\right)\left(x_3-x_2,e_{i_3}\right)},$$

$$a_{3,0,1}=\left[\frac{\left(x_2-x_1,e_{i_2}\right)}{\left(x_3-x_1,e_{i_2}\right)}\nabla_{i_1,i_2}F\left(x_{(i_1,i_2,i_2)}\right)-\nabla_{i_1,i_2}F\left(x_{(i_1,i_2,i_3)}\right)\Big|_{i_3=i_2+1}\right]\times$$

$$\times\left(x_1-x_0,e_{i_1}\right)^{-1}\left(x_2-x_1,e_{i_2}\right)^{-1}\left(x_3-x_2,e_{i_2}\right)^{-1},$$

$$a_{3,0,2}=\left[\frac{\left(x_1-x_0,e_{i_1}\right)}{\left(x_3-x_0,e_{i_1}\right)}\nabla_{i_1}F\left(x_{(i_1,i_1,i_1)}\right)-\nabla_{i_1}F\left(x_{(i_1,t,t)}\right)\Big|_{t=i_1+1}\right]\times$$

$$\times\left(x_1-x_0,e_{i_1}\right)^{-1}\left(x_3-x_1,e_{i_1}\right)^{-1}\left(x_3-x_2,e_{i_1}\right)^{-1},$$

$$a_{3,1,1}=\left[\frac{\left(x_1-x_0,e_{i_1}\right)}{\left(x_2-x_0,e_{i_1}\right)}\nabla_{i_1,i_2}F\left(x_{(i_1,i_1,i_2)}\right)-\nabla_{i_1,i_2}F\left(x_{(i_1,t,i_2)}\right)\Big|_{t=i_1+1}\right]\times$$

$$\times\left(x_1-x_0,e_{i_1}\right)^{-1}\left(x_2-x_1,e_{i_2}\right)^{-1}\left(x_3-x_2,e_{i_2}\right)^{-1},$$

$$a_{3,1,2} = \left[\nabla_{i_1} F\left(x_{(i_1,t,t)}\right)\Big|_{t=i_1+1} - \frac{\left(x_1 - x_0, e_{i_1}\right)}{\left(x_2 - x_0, e_{i_1}\right)} \nabla_{i_1} F\left(x_{(i_1,i_1,t)}\right)\Big|_{t=i_1+1} \right] \times$$

$$\times \left(x_1 - x_0, e_{i_1}\right)^{-1} \left(x_2 - x_1, e_{i_1}\right)^{-1} \left(x_3 - x_2, e_{i_1}\right)^{-1}. \quad (39)$$

The proof is analogous to the previous two cases.

The following theorem is valid.

Theorem 12. *Let the operator $F : H \to Y$ be such that for fixed x_0, x_1, x_2 and arbitrary x from H the following holds*

$$\sum_{i_1=1}^{\infty} \left\| a_{1,0,0} \right\|_Y < \infty, \quad \sum_{i_1=1}^{\infty} \sum_{i_2=i_1+1}^{\infty} \left\| a_{2,0,0} \right\|_Y < \infty, \quad \sum_{i_1=1}^{\infty} \left\| a_{2,0,1} \right\|_Y < \infty,$$

$$\sum_{i_1=1}^{\infty} \sum_{i_2=i_1+1}^{\infty} \sum_{i_3=i_2+1}^{\infty} \left\| a_{3,0,0} \right\|_Y < \infty, \quad \sum_{i_1=1}^{\infty} \sum_{i_2=i_1+1}^{\infty} \left\| a_{3,0,1} \right\|_Y < \infty, \quad \sum_{i_1=1}^{\infty} \left\| a_{3,0,2} \right\|_Y < \infty,$$

$$\sum_{i_1=1}^{\infty} \sum_{i_2=i_1+1}^{\infty} a_{3,1,1} < \infty, \quad \sum_{i_1=1}^{\infty} a_{3,1,2} < \infty.$$

Then the correct representation for this operator is $F(x) = P_2^I(x) + R_2(x)$, *where* $R_2(x) = p_3^I(x)\big|_{x_3=x}$.

The proof can be found in [5].

Remark 6. The interpolating operator polynomial (38), (39) is a Newton-type interpolation polynomial in the sense that when we transition from $P_n^I(x)$ to $P_{n+1}^I(x)$, we only need to add a new term to the polynomial $P_n^I(x)$.

 ISBN 978-3-98924-038-4

Summary and conclusions

We find a fundamentally new construction of interpolation operator polynomials with the properties of uniqueness and invariance for an abstract Banach space using a specially chosen countable sequence of nodes connected with the basis of the space.

For a certain Banach space $C[0,\infty)$ using a finite sequence of continuum nodes, a second construction with the same properties is found. Interpolation polynomials (of integral form) are also constructed, which also possess the property of uniqueness and invariance with respect to all integral polynomials of the same degree.

In a separable Hilbert space H, we prove the existence of a unique, invariant interpolation polynomial with a specially chosen countable set of interpolation nodes associated with the orthonormalized basis of the space H.

Necessary and sufficient conditions for the interpolativity of a Newton-type operator polynomial of the first, second, and third degrees on a countable set of interpolation nodes, which does not require the substitution rule, are obtained. The residual term of the interpolation polynomial of the second degree is found.

The publication was performed within the framework of the state budget theme of the Ministry of Education and Science of Ukraine No. 0123U101691.

Recommended for publication by the Academic Council of the Institute of Applied Mathematics and Fundamental Sciences of Lviv Polytechnic National University, Protocol No. 68 of 20.02.2024.