KAPITEL 12 / CHAPTER 12 [12]

QUALITATIVE METHODS IN MATHEMATICAL MODELING OF COMPLEX MECHANICAL SYSTEMS WITH DISSIPATION

DOI: 10.30890/2709-2313.2024-27-00-017

Introduction

This section of this monograph is devoted to the investigation of weakly and strongly nonlinear models of oscillatory systems that simulate the vibrations of unbounded bodies with account of dissipative and drag forces. These models are described by the vibration equation of a beam and equations called equations of the beam vibration equation type. In the model case, these equations have the form

$$\frac{\partial^2 u}{\partial t^2} - a\left|\frac{\partial^5 u}{\partial t \partial x^4}\right|^{\beta} \frac{\partial^5 u}{\partial t \partial x^4} + b\frac{\partial^4 u}{\partial x^4} + g_2\left(x,t,\frac{\partial u}{\partial t}\right) = f,$$

$$\frac{\partial^2 u}{\partial t^2} - a\frac{\partial^5 u}{\partial t \partial x^4} + b\frac{\partial^4 u}{\partial x^4} + c\left|\frac{\partial^3 u}{\partial t \partial x^2}\right|^{\beta} \frac{\partial^3 u}{\partial t \partial x^2} + g_2\left(x,t,\frac{\partial u}{\partial t}\right) = f,$$

$\beta > 0$. Some coefficients in these equations can increase powerfully as $x \to \infty$. The classes of existence and uniqueness (for some problems only existence) of the generalized solution without restrictions on the behavior of the solution at $x \to \infty$, the right-hand side of the equation, and the initial data are obtained. These classes are spaces of locally integrable functions.

In Section 2, the correctness classes of a solution to a mixed problem are certain weighted Sobolev spaces of functions that describe the qualitative behavior of the solution at infinity, which depends on the right-hand side of the equation and the initial data of the problem. The described research methodology allows us to obtain an estimate of the generalized solution of the problem. The results were published in [1].

The analysis of small transverse vibrations of a viscoelastic plane isotropic medium occupies an important place both in the theory of applied mechanics and in the theory of mathematical modeling of physical and mechanical processes. This problem often arises in industry, for example, when investigating vibrations of

[12] ***Authors:*** *Pukach Petro Yaroslavovych, Kunynets Andrii Volodymyrovych*

overhead line wires (in one-dimensional version), vibrations of oil tank walls in powerful energy transformers, vibrations of various platforms, etc. Such processes are described by the equations of small transverse vibrations taking into account external and internal dissipations of mechanical energy [2]. The linear analog of this equation was obtained in [2] on the basis of the modified Ostrogradsky-Hamilton integral variational principle using the extended Lagrange function.

12.1. Investigation of the mathematical model of oscillations of an unbounded body described by the equation of the type of oscillations of a beam with a perturbed linear operator

In this section, we consider the first mixed problem for the nonlinear equation

$$\mathbf{A}(u) \equiv \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(a_2(x,t)\frac{\partial^3 u}{\partial x^2 \partial t}\right) + \frac{\partial^2}{\partial x^2}\left(a^2(x,t)\left|\frac{\partial^3 u}{\partial x^2 \partial t}\right|^{p_2-2}\frac{\partial^3 u}{\partial x^2 \partial t}\right) -$$

$$-\frac{\partial}{\partial x}\left(a^1(x,t)\left|\frac{\partial^2 u}{\partial t \partial x}\right|^{p_1-2}\frac{\partial^2 u}{\partial t \partial x}\right) + a_0(x,t)\frac{\partial u}{\partial t} +$$

$$+\frac{\partial}{\partial x^2}\left(b_2(x)\frac{\partial^2 u}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(b_1(x)\frac{\partial u}{\partial x}\right) + b_0(x)u +$$

$$+c_2(x,t)\frac{\partial^2 u}{\partial x^2} + c_1(x,t)\frac{\partial u}{\partial x} + g(x,\frac{\partial u}{\partial t}) = f(x,t) \quad (1)$$

The equation of the form (1) generalizes the model of beam oscillation in a resisting medium, taking into account the influence of a wide range of physical, mechanical, kinematic, and elastic factors of both linear and (significantly) nonlinear nature on the oscillating system (see [3] and the cited bibliography).

In the domain $Q_T = (0,+\infty)\times(0,T)$, $0 < T < \infty$ we consider for equation (1) a mixed problem with initial conditions

$$u(x,0) = u_0(x), \quad (2)$$

$$\frac{\partial u(x,0)}{\partial t} = u_1(x) \qquad (3)$$

and boundary conditions

$$u(0,t) = \frac{\partial^2 u(0,t)}{\partial x^2} = 0 \, . \qquad (4)$$

We assume the following conditions for the coefficients, the right-hand side of equation (1), and the initial data.

(A) $\underset{(x,t)\in Q_T^R}{\mathbf{essup}}\left|a_2(x,t)\right| \le a_2^2 R^{\omega_2}$ for any $R>1$, where $0 \le \omega_2 < 1$, $a_2^2 > 0$; $a_2(x,t) \ge a_{0,2}$ for almost all $(x,t)\in Q_T$, where $a_{0,2} = \text{const} > 0$; the function $\frac{\partial a_2(x,t)}{\partial x^2}$ belongs to $L^\infty(Q_T)$; $a_0 \in L^\infty\left((0,T);L^\infty_{loc}(0,+\infty)\right)$, $a_0(x,t) \ge a_0$ for almost all $(x,t)\in Q_T$, where $a_0 = \text{const}$.

(A21) $a_2 \le a^2(x,t) \le a^2 R^{\alpha_2}$, $a_1 \le a^1(x,t) \le a^1 R^{\alpha_1}$ for arbitrary $R>1$ and for almost all $(x,t)\in Q_T^R$, where $0 \le \alpha_s < 1$, $a_s > 0$, $a^s > 0$, $\alpha_s \in \left[0;\frac{3p_s - 2}{2p_s}\right)$, $s = 1,2$; functions $\frac{\partial^2 a^2(x,t)}{\partial x^2}$, $\frac{\partial a^2(x,t)}{\partial t}$, $\frac{\partial a^1(x,t)}{\partial x}$, $\frac{\partial a^1(x,t)}{\partial t}$ belong to the space $L^\infty(Q_T)$.

(B) $\underset{x\in(0,R)}{\text{essup}}\left|b_2(x)\right| \le b^2 R^{\kappa_2}$ for any $R>1$, where $0 \le \kappa_2 < 1$, $b^2 > 0$; $b_2(x) \ge b_{0,2}$ for almost all $x \in (0,+\infty)$, $b_{0,2} > 0$; the functions $\frac{\partial^2 b_2(x)}{\partial x^2}$, $\frac{\partial b_1(x)}{\partial x}$, $b_0(x)$ belong to $L^\infty(0,+\infty)$.

(P21) $1 < p_2 < 2$, $1 < p_1 < 2$.

A generalized solution to problems (1)-(4) is a function

$$u \in C([0,T];H^2_{0,loc}(0,+\infty))$$

such that $\frac{\partial u}{\partial t} \in C\left([0,T];L^2_{loc}(0,+\infty)\right) \cap L^2\left((0,T);H^2_{0,loc}(0,+\infty)\right) \cap L^p\left((0,T);L^p_{loc}(0,+\infty)\right)$, which satisfies condition (2) and the integral identity

$$\int\limits_{Q_\tau}\left[-\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}+a_2(x,t)\frac{\partial^3 u}{\partial t\partial x^2}\frac{\partial^2 v}{\partial x^2}+a^2(x,t)\left|\frac{\partial^3 u}{\partial t\partial x^2}\right|^{p_2-2}\frac{\partial^3 u}{\partial t\partial x^2}\frac{\partial^2 v}{\partial x^2}\right]dxdt+$$

$$+\int\limits_{Q_\tau}\left[a_1(x,t)\left|\frac{\partial^2 u}{\partial t\partial x}\right|^{p_1-2}\frac{\partial^2 u}{\partial t\partial x}\frac{\partial v}{\partial x}+a_0(x,t)\frac{\partial u}{\partial t}v\right]dxdt+$$

$$+\int\limits_{Q_\tau}\left[b_2(x,t)\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2}+b_1(x,t)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+b_0(x,t)v\right]dxdt+$$

$$+\int\limits_{Q_\tau}\left[c_2(x,t)\frac{\partial^2 u}{\partial x^2}v+c_1(x,t)\frac{\partial u}{\partial x}v+g(x,\frac{\partial u}{\partial t})v-f(x,t)v\right]dxdt+$$

$$+\int\limits_{0}^{+\infty}\left[\frac{\partial u}{\partial t}(x,\tau)v(x,\tau)-u_1(x)v(x,0)\right]dx=0 \qquad (5)$$

for an arbitrary $\tau\in(0,T]$ and for an arbitrary function $v\in C^1([0,T];C_0^\infty(0,+\infty))$.

The main result of this section is the following statement.

Let conditions **(A), (A21), (B), (P21)** and conditions **(C), (G1), (F), (U)** of the previous paragraph be satisfied. Then there exists a unique generalized solution u to the problem (1) – (4) in Q_T.

Auxiliary result. We use the following auxiliary statement to prove the main result.

Let conditions **(A), (A21), (B), (P21)** of this paragraph and conditions **(C), (G1), (U)** of the previous paragraph be satisfied, u and u^0 be the generalized solutions of problems (1)-(4) and (1)-(4) for the equation $\mathrm{A}(u)=f^0$, $\frac{\partial\left(u-u^0\right)}{\partial t}\in L^2\left((0,T);L^2\left(0,R\right)\right)$ for any $R>1$. Then for arbitrary τ, R_0 such that $\tau\in(0,T]$, $1<R_0<R$, the correct estimate is

$$\int\limits_0^{R_0}\left[\left(\frac{\partial}{\partial t}\left(u(x,\tau)-u^0(x,\tau)\right)\right)^2+\left(\frac{\partial^2}{\partial x^2}\left(u(x,\tau)-u^0(x,\tau)\right)\right)^2\right]dx+$$

ISBN 978-3-98924-038-4

$$+\int_{Q_\tau^{R_0}}\left[\left(\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}-\frac{\partial u^0}{\partial t}\right)\right)^2+\left|\frac{\partial u}{\partial t}-\frac{\partial u^0}{\partial t}\right|^p\right]dxdt\le\left(\frac{R}{R-R_0}\right)^\gamma\times$$

$$\times\left[M_1R^{1-\beta}+M_2R^{1+(\alpha_2-1)\frac{2p_2}{2-p_2}}+M_3R^{1+(\alpha_1-1)\frac{2p_1}{2-p_1}}+M_4\int_{Q_\tau^R}\left|f-f^0\right|^{p'}dxdt\right] \tag{6}$$

where $\beta=\min\left\{(1-\omega_2)\frac{4q}{q-2};(1-\kappa_2)\frac{4q}{q-2};\frac{2q}{q-2}\right\}$, $q\in(2,p]$ is an arbitrary number, $\gamma>\frac{4q}{q-2}$, M_1, M_2, M_3, M_4 are positive constants that are independent of u, u^0, f, f^0.

The method of obtaining the auxiliary result. Let $R>1$, $\tau\in(0,T]$, γ be a sufficiently large positive number, the value of which will be specified below. We consider the truncating function $\phi_R(x)=\begin{cases}\frac{R^2-x^2}{R},x\le R,\\ 0,x>R.\end{cases}$ Let us denote $w=u-u^0$ and subtract from the integral equality (5) for the functions u, f the corresponding integral equality similar to (5) for u^0, f^0, by substituting $v=\frac{\partial w}{\partial t}\phi_R^\gamma$. Applying the conditions of the auxiliary statement, we obtain

$$\int_{Q_\tau^R}\left[\frac{\partial^2 w}{\partial t^2}\frac{\partial w}{\partial t}\phi_R^\gamma+a_2(x,t)\frac{\partial^3 w}{\partial x^2\partial t}\frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial t}\phi_R^\gamma\right)+b_2(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial t}\phi_R^\gamma\right)+\right.$$

$$+b_1(x)\frac{\partial w}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial t}\phi_R^\gamma\right)+b_0(x)w\frac{\partial w}{\partial t}\phi_R^\gamma+a_0(x,t)\left(\frac{\partial w}{\partial t}\right)^2\phi_R^\gamma$$

$$+a^2(x\ ,t)\left(\left|\frac{\partial^3 u}{\partial x^2\partial t}\right|^{p_2-2}\frac{\partial^3 u}{\partial x^2\partial t}-\left|\frac{\partial^3 u^0}{\partial x^2\partial t}\right|^{p_2-2}\frac{\partial^3 u^0}{\partial x^2\partial t}\right)\frac{\partial^2\left(\frac{\partial w}{\partial t}\phi_R^\gamma\right)}{\partial x^2}+$$

$$+a^1(x\ ,t)\left(\left|\frac{\partial^2 u}{\partial x\partial t}\right|^{p_1-2}\frac{\partial^2 u}{\partial x\partial t}-\left|\frac{\partial^2 u^0}{\partial x\partial t}\right|^{p_1-2}\frac{\partial^2 u^0}{\partial x\partial t}\right)\frac{\partial\left(\frac{\partial w}{\partial t}\phi_R^\gamma\right)}{\partial x}+$$

$$+\left[c_2(x,t)\frac{\partial^2 w}{\partial x^2}+c_1(x,t)\frac{\partial w}{\partial x}\right]\frac{\partial w}{\partial t}\phi_R^\gamma+\int_{Q_\tau^R}\left[\left(g(x,\frac{\partial u}{\partial t})-g(x,t,\frac{\partial u^0}{\partial t})\right)\times\right.$$

$$\left.\times\left(\frac{\partial u}{\partial t}-\frac{\partial u^0}{\partial t}\right)\phi_R^\gamma-\left(f-f^0\right)\frac{\partial w}{\partial t}\phi_R^\gamma\right]dxdt=0 \quad . \qquad (4.7)$$

Since $\frac{\partial w}{\partial t}(x,0)=0$ it holds for almost all $x\in(0,+\infty)$, then

$$\int_{Q_\tau^R}\frac{\partial^2 w}{\partial t^2}\frac{\partial w}{\partial t}\phi_R^\gamma dxdt=\frac{1}{2}\int_0^R\left(\frac{\partial w(x,\tau)}{\partial t}\right)^2\phi_R^\gamma dx.$$

Moreover, we obtain

$$\int_{Q_\tau^R}a_2(x,t)\frac{\partial^3 w}{\partial x^2\partial t}\frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial t}\phi_R^\gamma\right)dxdt=\int_{Q_\tau^R}a_2(x,t)\left(\frac{\partial^3 w}{\partial x^2\partial t}\right)^2\phi_R^\gamma dxdt+$$

$$+2\int_{Q_\tau^R}a_2(x,t)\frac{\partial^3 w}{\partial x^2\partial t}\frac{\partial^2 w}{\partial x\partial t}\frac{\partial\phi_R^\gamma}{\partial x}dxdt+\int_{Q_\tau^R}a_2(x,t)\frac{\partial^3 w}{\partial x^2\partial t}\frac{\partial w}{\partial t}\frac{\partial^2\phi_R^\gamma}{\partial x^2}dxdt=I_1+I_2+I_3.$$

The integrals I_1, I_2, I_3 are estimated as follows:

$$I_1\ge a_{0,2}\int_{Q_\tau^R}\left(\frac{\partial^3 w}{\partial x^2\partial t}\right)^2\phi_R^\gamma dxdt;$$

$$|I_2|\le(a_2\delta_1+C_1\delta_2)\int_{Q_\tau^R}\left(\frac{\partial^3 w}{\partial x^2\partial t}\right)^2\phi_R^\gamma dxdt+C_2\delta_3\int_{Q_\tau^R}\left|\frac{\partial w}{\partial t}\right|^p\phi_R^\gamma dxdt+$$

$$+C_2\delta_3\int_{Q_\tau^R}\left(\frac{\partial w}{\partial t}\right)^2\phi_R^\gamma dxdt+C_2C_3R^{\gamma+1+(\omega_2-1)\frac{4q}{q-2}},$$

where $q\in(2,p]$, $\gamma>\frac{4q}{q-2}$, $\delta_1>0$, $\delta_2>0$, $\delta_3>0$ are arbitrary numbers, C_1 is a positive constant that depends on γ, δ_1, a_2^2, $C_2>0$ is a constant that depends on γ,

a_2^2, δ_1, C_3 is a positive constant that depends on γ, T, δ_3;

$$|I_3| \le \delta_4 \int\limits_{Q_\tau^R} \left(\frac{\partial^3 w}{\partial x^2 \partial t} \right)^2 \phi_R^\gamma dxdt + C_4 \delta_4 \int\limits_{Q_\tau^R} \left| \frac{\partial w}{\partial t} \right|^p \phi_R^\gamma dxdt +$$

$$+ C_4 \delta_4 \int\limits_{Q_\tau^R} \left(\frac{\partial w}{\partial t} \right)^2 \phi_R^\gamma dxdt + C_5 R^{\gamma + 1 + (\omega_2 - 2)\frac{2q}{q-2}},$$

where $\delta_4 > 0$ is an arbitrary constant, $C_4 > 0$ is a constant that depends on γ, δ_4, a_2^2, C_5 is a positive constant that depends on γ, T, δ_4, a_2^2.

Now let us transform it further

$$\int\limits_{Q_\tau^R} a^2(x,t) \left(\left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^{p_2 - 2} \frac{\partial^3 u}{\partial x^2 \partial t} - \left| \frac{\partial^3 u^0}{\partial x^2 \partial t} \right|^{p_2 - 2} \frac{\partial^3 u^0}{\partial x^2 \partial t} \right) \times$$

$$\times \left(\frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial^3 u^0}{\partial x^2 \partial t} \right) \phi_R^\gamma dxdt = \int\limits_{Q_\tau^R} a^2(x,t) \left(\left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^{p_2 - 2} \frac{\partial^3 u}{\partial x^2 \partial t} - \right.$$

$$\left. - \left| \frac{\partial^3 u^0}{\partial x^2 \partial t} \right|^{p_2 - 2} \frac{\partial^3 u^0}{\partial x^2 \partial t} \right) \left(\frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial^3 u^0}{\partial x^2 \partial t} \right) \phi_R^\gamma dxdt + 2 \int\limits_{Q_\tau^R} a^2(x,t) \times$$

$$\times \left(\left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^{p_2 - 2} \frac{\partial^3 u}{\partial x^2 \partial t} - \left| \frac{\partial^3 u^0}{\partial x^2 \partial t} \right|^{p_2 - 2} \frac{\partial^3 u^0}{\partial x^2 \partial t} \right) \left(\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u^0}{\partial x \partial t} \right) \frac{\partial \phi_R^\gamma}{\partial x} dxdt +$$

$$+ \int\limits_{Q_\tau^R} a^2(x,t) \left(\left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^{p_2 - 2} \frac{\partial^3 u}{\partial x^2 \partial t} - \left| \frac{\partial^3 u^0}{\partial x^2 \partial t} \right|^{p_2 - 2} \frac{\partial^3 u^0}{\partial x^2 \partial t} \right) \left(\frac{\partial u}{\partial t} - \frac{\partial u^0}{\partial t} \right) \times \frac{\partial^2 \phi_R^\gamma}{\partial x^2} dxdt = I_4 + I_5 + I_6.$$

Considering the conditions **(A21), (P21)** the integrals I_4, I_5, I_6 are estimated as follows:

$$I_4 \ge a_2 \int\limits_{Q_\tau^R} \left(\left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^{p_2 - 2} \frac{\partial^3 u}{\partial x^2 \partial t} - \left| \frac{\partial^3 u^0}{\partial x^2 \partial t} \right|^{p_2 - 2} \frac{\partial^3 u^0}{\partial x^2 \partial t} \right) \times \left(\frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial^3 u^0}{\partial x^2 \partial t} \right) \phi_R^\gamma dxdt \ge 0;$$

$$I_5 \ge - \delta_5 \int\limits_{Q_\tau^R} \left| \left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^{p_2 - 2} \frac{\partial^3 u}{\partial x^2 \partial t} - \left| \frac{\partial^3 u^0}{\partial x^2 \partial t} \right|^{p_2 - 2} \frac{\partial^3 u^0}{\partial x^2 \partial t} \right|^{p_2'} \phi_R^\gamma dxdt -$$

 ISBN 978-3-98924-038-4

$$-\delta_5 \int_{Q_\tau^R} \left(\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u^0}{\partial x \partial t} \right)^2 \phi_R^\gamma dxdt - C_6 \int_{Q_\tau^R} \left| a^2(x,t) \phi_R^{\gamma - 1 - \frac{\gamma}{p_{2'}} - \frac{\gamma}{2}} \right|^{\frac{2p_2}{2-p_2}} \times$$

$$\times dxdt \geq -2^{2-p_2} \delta_5 \int_{Q_\tau^R} \left(\left| \frac{\partial^2 u}{\partial x \partial t} \right|^{p_2 - 2} \frac{\partial^2 u}{\partial x \partial t} - \left| \frac{\partial^2 u}{\partial x \partial t} \right|^{p_2 - 2} \frac{\partial^2 u}{\partial x \partial t} \right) \times$$

$$\times \left(\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u^0}{\partial x \partial t} \right) \phi_R^\gamma dxdt - \delta_5 \int_{Q_\tau^R} \left(\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u^0}{\partial x \partial t} \right)^2 \phi_R^\gamma dxdt - C_7(\delta_5, \gamma, a_2, T) R^{\gamma + 1 + \frac{2p_2(\alpha_2 - 1)}{2 - p_2}},$$

where $\delta_5 > 0$ is an arbitrary constant, $C_6 > 0$, $C_7 > 0$;

$$I_6 \geq -2^{2-p_2} \delta_6 \int_{Q_\tau^R} \left(\left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^{p_2 - 2} \frac{\partial^3 u}{\partial x^2 \partial t} - \left| \frac{\partial^3 u^0}{\partial x^2 \partial t} \right|^{p_2 - 2} \frac{\partial^3 u^0}{\partial x^2 \partial t} \right) \times$$

$$\times \left(\frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial^3 u^0}{\partial x^2 \partial t} \right) \phi_R^\gamma dxdt - \delta_6 T \int_0^R \left(\frac{\partial u(x,\tau)}{\partial t} - \frac{\partial u^0(x,\tau)}{\partial t} \right)^2 \phi_R^\gamma dx -$$

$$- C_8(\delta_6, \gamma, a_2, T) R^{\gamma + 1 + \frac{2p_2(\alpha_2 - 2)}{2 - p_2}},$$

$\delta_6 > 0$ is an arbitrary constant, $C_8 > 0$. Let us continue to transform and estimate the integrals of equality (7). We obtain

$$\int_{Q_\tau^R} a_1(x,t) \left(\left| \frac{\partial^2 u}{\partial x \partial t} \right|^{p_1 - 2} \frac{\partial^2 u}{\partial x \partial t} - \left| \frac{\partial^2 u^0}{\partial x \partial t} \right|^{p_1 - 2} \frac{\partial^2 u^0}{\partial x \partial t} \right) \times$$

$$\times \frac{\partial}{\partial x} \left(\left(\frac{\partial u}{\partial t} - \frac{\partial u^0}{\partial t} \right) \phi_R^\gamma \right) dxdt = \int_{Q_\tau^R} a_1(x,t) \left(\left| \frac{\partial^2 u}{\partial x \partial t} \right|^{p_1 - 2} \frac{\partial^2 u}{\partial x \partial t} - \right.$$

$$\left. - \left| \frac{\partial^2 u^0}{\partial x \partial t} \right|^{p_1 - 2} \frac{\partial^2 u^0}{\partial x \partial t} \right) \left(\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u^0}{\partial x \partial t} \right) \phi_R^\gamma dxdt + \int_{Q_\tau^R} a_1(x,t) \times$$

$$\times \left(\left| \frac{\partial^2 u}{\partial x \partial t} \right|^{p_1 - 2} \frac{\partial^2 u}{\partial x \partial t} - \left| \frac{\partial^2 u^0}{\partial x \partial t} \right|^{p_1 - 2} \frac{\partial^2 u^0}{\partial x \partial t} \right) \left(\frac{\partial u}{\partial t} - \frac{\partial u^0}{\partial t} \right) \frac{\partial \phi_R^\gamma}{\partial x} dxdt = I_7 + I_8.$$

Let us estimate the integrals I_7, I_8 as follows:

 ISBN 978-3-98924-038-4

$$I_7 \ge a_1 \int_{Q_\tau^R} \left(\left| \frac{\partial^2 u}{\partial x \partial t} \right|^{p_1-2} \frac{\partial^2 u}{\partial x \partial t} - \left| \frac{\partial^2 u^0}{\partial x \partial t} \right|^{p_1-2} \frac{\partial^2 u^0}{\partial x \partial t} \right) \left(\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u^0}{\partial x \partial t} \right) \phi_R^\gamma dxdt \ge 0;$$

$$I_8 \ge -2^{2-p_2} \delta_7 \int_{Q_\tau^R} \left(\left| \frac{\partial^2 u}{\partial x \partial t} \right|^{p_1-2} \frac{\partial^2 u}{\partial x \partial t} - \left| \frac{\partial^2 u^0}{\partial x \partial t} \right|^{p_1-2} \frac{\partial^2 u^0}{\partial x \partial t} \right) \times$$

$$\times \left(\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u^0}{\partial x \partial t} \right) \phi_R^\gamma dxdt - \delta_7 \int_{Q_\tau^R} \left(\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u^0}{\partial x \partial t} \right)^2 \phi_R^\gamma dxdt -$$

$$-C_9(\delta_7, \gamma, a_1, T) R^{\gamma + 1 + \frac{2p_1(\alpha_1 - 1)}{2 - p_1}},$$

$\delta_7 > 0$ is an arbitrary constant, $C_9 > 0$. Applying Bernis' lemma [4], we estimate

$$(\delta_5 + \delta_7) \int_{Q_\tau^R} \left(\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u^0}{\partial x \partial t} \right)^2 \phi_R^\gamma dxdt \le (\delta_5 + \delta_7)\delta_8 \int_{Q_\tau^R} \left(\frac{\partial^3 w}{\partial x^2 \partial t} \right)^2 \phi_R^\gamma dxdt +$$

$$+ \frac{\delta_5 + \delta_7}{\delta_8} \int_{Q_\tau^R} \left(\frac{\partial w}{\partial t} \right)^2 \phi_R^\gamma dxdt + C_{10}(\delta_5, \delta_7) \int_{Q_\tau^R} \left(\frac{\partial w}{\partial t} \right)^2 \phi_R^{\gamma-2} dxdt \le (\delta_5 + \delta_7) \times$$

$$\times \delta_8 \int_{Q_\tau^R} \left(\frac{\partial^3 w}{\partial x^2 \partial t} \right)^2 \phi_R^\gamma dxdt + \frac{\delta_5 + \delta_7}{\delta_8} \int_{Q_\tau^R} \left(\frac{\partial w}{\partial t} \right)^2 \phi_R^\gamma dxdt + C_{10}(\delta_5, \delta_7) \times$$

$$\times \delta_9 \left[\int_{Q_\tau^R} \left| \frac{\partial w}{\partial t} \right|^p \phi_R^\gamma dxdt + \delta_9 \int_{Q_\tau^R} \left(\frac{\partial w}{\partial t} \right)^2 \phi_R^\gamma dxdt + C_{11}(\delta_9) R^{\gamma + 1 + \frac{2q}{q-2}} \right] =$$

$$= (\delta_5 + \delta_7)\delta_8 \int_{Q_\tau^R} \left(\frac{\partial^3 w}{\partial x^2 \partial t} \right)^2 \phi_R^\gamma dxdt + \left(\frac{\delta_5 + \delta_7}{\delta_8} + C_{10}\delta_9 \right) \int_{Q_\tau^R} \left(\frac{\partial w}{\partial t} \right)^2 \phi_R^\gamma dxdt +$$

$$+ C_{10}\delta_9 \int_{Q_\tau^R} \left| \frac{\partial w}{\partial t} \right|^p \phi_R^\gamma dxdt + C_{12}(\gamma, T, \delta_5, \delta_7, \delta_9) R^{\gamma + 1 + \frac{2q}{q-2}},$$

where $\delta_8 > 0$, $\delta_9 > 0$ are arbitrary constants, $C_{10} > 0$, $C_{11} > 0$, $C_{12} > 0$.

Taking into account these and the above estimates of the integrals of equality (7) and consistently choosing sufficiently small positive constants $\delta_1, \delta_2, \dots$, we obtain, after appropriate redefinitions, that

 ISBN 978-3-98924-038-4

$$\int_0^R \left[\left(\frac{\partial w}{\partial t}(x,\tau) \right)^2 + \left(\frac{\partial^2 w}{\partial x^2}(x,\tau) \right)^2 \right] \phi_R^\gamma dx + \int_{Q_\tau^R} \left(\frac{\partial^3 w}{\partial x^2 \partial t} \right)^2 \phi_R^\gamma dxdt +$$

$$+ \int_{Q_\tau^R} \left| \frac{\partial w}{\partial t} \right|^p \phi_R^\gamma dxdt \le C_{13} \int_{Q_\tau^R} \left(\frac{\partial w}{\partial t} \right)^2 \phi_R^\gamma dxdt + C_{14} \int_{Q_\tau^R} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \phi_R^\gamma dxdt +$$

$$+ C_{15} R^{\gamma+1-\beta} + C_{16} R^{\gamma+1+(\alpha_2-1)\frac{2p_2}{2-p_2}} +$$

$$+ C_{17} R^{\gamma+1+(\alpha_1-1)\frac{2p_1}{2-p_1}} + C_{18} \int_{Q_\tau^R} \left| f - f^0 \right|^{p'} dxdt,$$

where the positive constants C_{13} – C_{18} are independent of R. Using Gronwall's lemma and taking into account the properties of the function $\phi_R(x)$, we obtain

$$(R-R_0)^\gamma \left[\int_0^{R_0} \left[\left(\frac{\partial w}{\partial t}(x,\tau) \right)^2 + \left(\frac{\partial^2 w}{\partial x^2}(x,\tau) \right)^2 \right] \phi_R^\gamma dx + \int_{Q_\tau^{R_0}} \left(\frac{\partial^3 w}{\partial x^2 \partial t} \right)^2 \phi_R^\gamma dxdt + \right.$$

$$\left. + \int_{Q_\tau^{R_0}} \left| \frac{\partial w}{\partial t} \right|^p \phi_R^\gamma dxdt \right] \le C_{19} R^{\gamma+1-\beta} + C_{20} R^{\gamma+1+(\alpha_2-1)\frac{2p_2}{2-p_2}} +$$

$$+ C_{21} R^{\gamma+1+(\alpha_1-1)\frac{2p_1}{2-p_1}} + C_{22} \int_{Q_\tau^R} \left| f - f^0 \right|^{p'} dxdt, \tag{8}$$

for any τ, R, R_0 such that $1 < R_0 < R$, $\tau \in (0,T]$, where C_{19} - C_{22} are positive constants independent of u, u^0, f, f^0. From inequality (8), it is simple to obtain (6).

Methodology for obtaining the main result. *Existence.* Let us choose a sequence of intervals $\{(0,k)\}$, $k = 2,3,...$ Consider an auxiliary problem in a bounded domain Q_T^k:

$$\mathbf{A}(u) = f^k(x,t), \tag{9}$$

$$u(x,0) = u_0^k(x), \tag{10}$$

$$\frac{\partial u(x,0)}{\partial t} = u_1^k(x), \quad x \in (0,k), \tag{11}$$

$$u(0,t) = \frac{\partial^2 u(0,t)}{\partial x^2} = 0, \; u(k,t) = \frac{\partial^2 u(k,t)}{\partial x^2} = 0. \tag{12}$$

Note that the functions u_0^k, u_1^k, f^k in this problem are chosen as follows

$$f^k \in C^1([0,T];C_0^1(0,+\infty)),\; f^k \to f \text{ в } L^2((0,T);L_{loc}^2(0,+\infty)),$$

$$u_0^k \in C_0^4(0,+\infty), \quad u_1^k \in C_0^2(0,+\infty),\; u_0^k \to u_0 \text{ в } H_{0,loc}^2(0,+\infty),$$

$$u_1^k \to u_1 \text{ в } H_{0,loc}^2(0/+\infty) \cap L_{loc}^{2p-2}(0,+\infty) \text{ при } k \to \infty.$$

On the basis of [5, pp. 67-68], [6, pp. 52-53] it can be stated that there is a unique generalized solution $u^k \in C([0,T];H_0^2(0,k))$ to problem (9) – (12) in the domain Q_T^k such that

$$\frac{\partial u^k}{\partial t} \in C\big([0,T];L^2(0,k)\big) \cap L^\infty\big((0,T);H_0^2(0,k)\big) \cap L^p\big(Q_T^k\big),$$

$$\frac{\partial^2 u^k}{\partial t^2} \in L^\infty\big((0,T);L^2(0,k)\big) \cap L^2\big((0,T);H_0^2(0,k)\big).$$

Now we consider the sequence of problems of the form (1) – (4) for $k=2$, $k=3$, ..., setting u^k zero at $Q_T \setminus Q_T^k$. We show that the sequence $\{u^k\}$ is fundamental in the space $C([0,T];H_{0,loc}^2(0,+\infty))$, and the sequence $\left\{\frac{\partial u^k}{\partial t}\right\}$ is fundamental in the space $C\big([0,T],L_{loc}^2(0,+\infty)\big) \cap L^2\big((0,T),H_{0,loc}^2(0,+\infty)\big) \cap L^p\big((0,T),L_{0,loc}^p(0,+\infty)\big)$.

In the domain Q_τ^R, where $R > R_0$, we consider the difference of $u^l - u^m$, and $l,m \in \mathbf{N}$, and apply the auxiliary result. Similarly to (6), we obtain

$$\int_0^{R_0} \left[\left(\frac{\partial}{\partial t}\big(u^l(x,\tau) - u^m(x,\tau)\big) \right)^2 + \left(\frac{\partial^2}{\partial x^2}\big(u^l(x,\tau) - u^m(x,\tau)\big) \right)^2 \right] dx +$$

$$+ \int_{Q_\tau^{R_0}} \left[\left(\frac{\partial^2}{\partial x^2}\left(\frac{\partial u^l}{\partial t} - \frac{\partial u^m}{\partial t} \right) \right)^2 + \left| \frac{\partial u^l}{\partial t} - \frac{\partial u^m}{\partial t} \right|^p \right] dxdt \le \left(\frac{R}{R-R_0} \right)^\gamma \times$$

 ISBN 978-3-98924-038-4

$$\times C_{23}R^{1-\beta}+\left(\frac{R}{R-R_0}\right)^{\gamma}\left(C_{24}R^{1+(\alpha_2-1)\frac{2p_2}{2-p_2}}+C_{25}R^{1+(\alpha_1-1)\frac{2p_1}{2-p_1}}\right)+$$

$$+C_{26}(R)\left\|u_0^l-u_0^m\right\|^2_{H_0^2\left(0,+\infty\right)}+\left(\frac{R}{R-R_0}\right)^{\gamma}C_{27}\left\|u_1^l-u_1^m\right\|^2_{L^2\left(0,+\infty\right)}+$$

$$+\left(\frac{R}{R-R_0}\right)^{\gamma}C_{28}\left\|f^l-f^m\right\|^{p'}_{L^{p'}\left((0,T);L^{p'}(0,+\infty)\right)},\tag{13}$$

where C_{23} - C_{28} – are some positive constants that depend on p, γ, q and the coefficients of (1). Let $\varepsilon>0$ – be an arbitrarily small number. Since $\lim\limits_{q\to 2+0}\frac{q}{q-2}=+\infty$, there exists such β_0, that $1<\beta_0\le\beta$. Since $\lim\limits_{R\to+\infty}\left(\frac{R}{R-R_0}\right)^{\gamma}=1$, there exists such $R_1>R_0$, that $\left(\frac{R}{R-R_0}\right)^{\gamma}C_{23}R^{1-\beta_0}<\frac{\varepsilon}{4}.$ Note that from the condition **(A21)** follows the existence of a sufficiently large positive number (without restriction of generality, we denote it as R_1), that

$$\left(\frac{R}{R-R_0}\right)^{\gamma}\left(C_{24}R^{1+(\alpha_2-1)\frac{2p_2}{2-p_2}}+C_{25}R^{1+(\alpha_1-1)\frac{2p_1}{2-p_1}}\right)<\frac{\varepsilon}{4}.$$

Based on the convergence of the sequences $\{f^k\}$, $\{u_0^k\}$ and $\{u_1^k\}$ in the corresponding function spaces, we can choose such $k_0\in\mathbf{N}$, $k_0>[R_1]+1$, that for all $l,m>k_0$ is correct estimate

$$C_{26}(R_1)\left\|u_0^l-u_0^m\right\|^2_{H_0^2\left(0,R^1\right)}+\left(\frac{R_1}{R_1-R_0}\right)^{\gamma}\times$$

$$\times\left(C_{27}\left\|u_1^l-u_1^m\right\|^2_{L^2\left(0,R_1\right)}+C_{28}\left\|f^l-f^m\right\|^{p'}_{L^{p'}\left((0,T);L^{p'}(0,R_1)\right)}\right)<\frac{\varepsilon}{2}.$$

Therefore, it follows from (13) that for any fixed $R_0>1$ and arbitrarily small value

$\varepsilon > 0$ there exists $k_0(\varepsilon) \in \mathbf{N}$, that

$$\int_0^{R_0} \left[\left(\frac{\partial}{\partial t}\left(u^l(x,\tau) - u^m(x,\tau)\right) \right)^2 + \left(\frac{\partial^2}{\partial x^2}\left(u^l(x,\tau) - u^m(x,\tau)\right) \right)^2 \right] dx +$$

$$+ \int_{Q_\tau^{R_0}} \left[\left(\frac{\partial^2}{\partial x^2}\left(\frac{\partial u^l}{\partial t} - \frac{\partial u^m}{\partial t} \right) \right)^2 + \left| \frac{\partial u^l}{\partial t} - \frac{\partial u^m}{\partial t} \right|^p \right] dxdt < \varepsilon$$

for any $l, m > k_0$, $\tau \in [0,T]$. Thus, the sequence $\{u^k\}$ is fundamental in the space $C([0,T]; H^2_{0,loc}(0,+\infty))$, and the sequence $\left\{ \frac{\partial u^k}{\partial t} \right\}$ is fundamental in the space $C\left([0,T], L^2_{loc}(0,+\infty)\right) \cap L^2\left((0,T), H^2_{0,loc}(0,+\infty)\right) \cap L^p\left((0,T), L^p_{0,loc}(0,+\infty)\right)$. Taking into account the arbitrariness of R_0, we obtain that the sequence $\{u^k\}$ converges to u in $C([0,T]; H^2_{0,loc}(0,+\infty))$, and the sequence $\left\{ \frac{\partial u^k}{\partial t} \right\}$ converges to $\frac{\partial u}{\partial t}$ in $C\left([0,T], L^2_{loc}(0,+\infty)\right) \cap L^2\left((0,T), H^2_{0,loc}(0,+\infty)\right) \cap L^p\left((0,T), L^p_{0,loc}(0,+\infty)\right)$. Moreover, conditions (2), (3), and (4) are satisfied for the function u. In addition, it is easy to obtain from the strong convergence $\left\{ \frac{\partial u^k}{\partial t} \right\}$ to $\frac{\partial u}{\partial t}$ that $g\left(x, \frac{\partial u^k}{\partial t}\right) \to g\left(x, \frac{\partial u}{\partial t}\right)$ is weak in $L^{p'}((0,T); L^{p'}_{loc}(0,+\infty))$ [7, p. 25, Lemma 1.3]. Note that, similarly to inequality (6), we can obtain estimates of

$$\int_{Q_\tau^{R_0}} \left| \frac{\partial^3 u^k}{\partial x^2 \partial t} \right|^{p_2} dxdt \le C_{29}, C_{29} > 0, \quad \int_{Q_\tau^{R_0}} \left| \frac{\partial^2 u^k}{\partial x \partial t} \right|^{p_1} dxdt \le C_{30}, C_{30} > 0,$$

from which, based on strong convergence, we find

$$\frac{\partial^2}{\partial x^2}\left(\left| \frac{\partial^3 u^k}{\partial x^2 \partial t} \right|^{p_2-2} \frac{\partial^3 u^k}{\partial x^2 \partial t} \right) \to \frac{\partial^2}{\partial x^2}\left(\left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^{p_2-2} \frac{\partial^3 u}{\partial x^2 \partial t} \right)$$ weakly in $L^{p'_2}\left((0,T); V_2^*\right)$,

 ISBN 978-3-98924-038-4

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial^2 u^k}{\partial x\partial t}\right|^{p_1-2}\frac{\partial^2 u^k}{\partial x\partial t}\right)\to\frac{\partial}{\partial x}\left(\left|\frac{\partial^2 u}{\partial x\partial t}\right|^{p_1-2}\frac{\partial^2 u}{\partial x\partial t}\right)$$ weakly in $L^{p_1'}\left((0,T);V_1^*\right)$,

where $V_2=W_{loc}^{2,p_2}\left(0,+\infty\right)$, $V_1=W_{loc}^{1,p_1}\left(0,+\infty\right)$. Thus, u is a generalized solution to problem (1) – (4).

Uniqueness. We consider two arbitrary solutions u^1 and u^2 problem (1) – (4). Similar to (6), we obtain an estimate

$$I\equiv\int_0^{R_0}\left[\left(\frac{\partial}{\partial t}\left(u^1(x,\tau)-u^2(x,\tau)\right)\right)^2+\left(\frac{\partial^2}{\partial x^2}\left(u^1(x,\tau)-u^2(x,\tau)\right)\right)^2\right]dx+$$

$$+\int_{Q_\tau^{R_0}}\left[\left(\frac{\partial^2}{\partial x^2}\left(\frac{\partial u^1}{\partial t}-\frac{\partial u^2}{\partial t}\right)\right)^2+\left|\frac{\partial u^1}{\partial t}-\frac{\partial u^2}{\partial t}\right|^p\right]dxdt\le\left(\frac{R}{R-R_0}\right)^\gamma\times$$

$$\times M_5R^{1-\beta}+M_6R^{1+(\alpha_2-1)\frac{2p_2}{2-p_2}}+M_7R^{1+(\alpha_1-1)\frac{2p_1}{2-p_1}}\Bigg],\qquad(14)$$

for any τ, R, R_0 such that $1<R_0<R$, $\tau\in[0,T]$. Similarly, as in the proof of fundamentality, we obtain by directing $R\to+\infty$, that $I\le0$ for any fixed $R_0>1$. Consequently, $u^1=u^2$ almost everywhere in $Q_T^{R_0}$. The arbitrariness of R_0 completes the proof of uniqueness.

12.2. Weight classes of solution correctness in a mathematical model of nonlinear vibrations of an unbounded beam with consideration of dissipation

In this section, the correctness classes of a solution are certain weighted Sobolev spaces of functions to a mixed problem in a mathematical model of nonlinear vibrations of an unbounded beam concerning dissipation that describe the qualitative behavior of the solution at infinity, which depends on the right-hand side of the equation and the initial data of the problem.

We investigate the first mixed problem for the weakly nonlinear equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(a_2(x,t)\frac{\partial^3 u}{\partial x^2\partial t}\right)-\frac{\partial}{\partial x}\left(a_1(x,t)\frac{\partial u}{\partial x\partial t}\right)+$$

$$+a_0(x,t)\frac{\partial u}{\partial t}+\frac{\partial}{\partial x^2}\left(b_2(x,t)\frac{\partial^2 u}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(b_1(x,t)\frac{\partial u}{\partial x}\right)+b_0(x,t)u+$$

$$+c_2(x,t)\frac{\partial^2 u}{\partial x^2}+c_1(x,t)\frac{\partial u}{\partial x}+g(x,t,\frac{\partial u}{\partial t})=f(x,t), \qquad (15)$$

in an unbounded spatially variable domain $Q_T=(0,+\infty)\times(0,T)$, $0<T<\infty$. We consider a mixed problem for equation (15) with initial conditions

$$u(x,0)=u_0(x), \qquad (16)$$

$$\frac{\partial u(x,0)}{\partial t}=u_1(x) \qquad (17)$$

and boundary conditions

$$u(0,t)=\frac{\partial^2 u(0,t)}{\partial x^2}=0. \qquad (18)$$

The function ψ has the following properties:

(Ψ) the function $\psi\in C^2(\mathbf{R})$ is monotonic at $x\to+\infty$, $\psi:[0,+\infty)\to(0,+\infty)$, $|\psi'(x)|\le M_1\psi(x)$, $M_1>0$, $|\psi''(x)|\le M_2\psi(x)$, $M_2>0$.

Remarks. Examples of functions ψ are $\psi(x)=(1+x)^\alpha$, $\alpha=\text{const}$, $\psi(x)=e^{\beta x}$, $\beta=\text{const}$ etc.

We use spaces with a weight function ψ:

$$L^{r,\psi}(0,+\infty)=\left\{u:\int_0^{+\infty}|u|^r\psi(x)dx<+\infty\right\},\quad r\in(1,+\infty),$$

$$\|u\|_{L^{r,\psi}(0,+\infty)}=\left(\int_0^{+\infty}|u|^r\psi(x)dx\right)^{1/r},$$

$H_0^{2,\psi}(0,+\infty)$ – closure of the space of infinitely differentiable functions in the domain

$(0,+\infty)$ with compact norm

$$\|u\|_{H_0^{2,\psi}(0,+\infty)} = \left(\int_0^{+\infty} \left[\left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \left(\frac{\partial u}{\partial x} \right)^2 + u^2 \right] \psi(x) dx \right)^{\frac{1}{2}}.$$

We assume the following conditions for the coefficients, the right-hand side of equation (15), and the initial data.

(A) The functions $a_2(x,t)$, $a_1(x,t)$, $a_0(x,t)$ belong to the space $L^\infty(Q_T)$; $a_2(x,t) \geq a_{0,2}$ for almost all $(x,t) \in Q_T$, $a_{0,2} > 0$.

(B) The functions $b_2(x,t)$, $\frac{\partial b_2(x,t)}{\partial t}$, $b_1(x,t)$, $b_0(x,t)$ belong to the space $L^\infty(Q_T)$; $b_2(x,t) \geq b_{0,2}$ for almost all $(x,t) \in Q_T$, $b_{0,2} > 0$.

(C) The functions $c_2(x,t)$, $c_1(x,t)$ belong to $L^\infty(Q_T)$.

(G) The function $g(x,t,\eta)$ – is measurable by x, continuous by t,η, and for any $\xi,\eta \in \mathbf{R}$ and for almost all $(x,t) \in (0;+\infty) \times (0,T)$ the conditions hold: $(g(x,t,\xi) - g(x,t,\eta))(\xi - \eta) \geq g_0 |\xi - \eta|^p$, $g_0 = \text{const} > 0$, $|g(x,t,\eta)| \leq g_1 |\eta|^{p-1}$, $g_1 = \text{const} > 0$, $p > 2$.

(F) $f \in L^{p'}((0,T); L^{p',\psi}(0,+\infty))$, $p' = p/(p-1)$.

(U) $u_0 \in H_0^{2,\psi}(0,+\infty)$, $u_1 \in L^{2,\psi}(0,+\infty)$.

A generalized solution to problems (15)-(18) is a function $u \in C([0,T]; H_0^{2,\psi}(0,+\infty))$ such that $\frac{\partial u}{\partial t} \in C\left([0,T]; \left(H_0^{2,\psi}(0,+\infty) \cap L^{p,\psi}(0,+\infty)\right)^*\right) \cap L^2\left((0,T); H_0^{2,\psi}(0,+\infty)\right) \cap L^p\left((0,T); L^{p,\psi}(0,+\infty)\right)$, which satisfies conditions (16), (18) and the integral identity

$$\int_{Q_\tau} \left[-\frac{\partial u}{\partial t} \frac{\partial v}{\partial t} \psi(x) + a_2(x,t) \frac{\partial^3 u}{\partial t \partial x^2} \frac{\partial^2 (v\psi(x))}{\partial x^2} \right] dxdt +$$

$$+\int_{Q_\tau}\left[a_1(x,t)\frac{\partial^2 u}{\partial t\partial x}\frac{\partial\left(v\psi(x)\right)}{\partial x}+a_0(x,t)\frac{\partial u}{\partial t}v\psi(x)+b_2(x,t)\frac{\partial^2 u}{\partial x^2}\frac{\partial^2\left(v\psi(x)\right)}{\partial x^2}\right]dxdt+$$

$$+\int_{Q_\tau}\left[b_1(x,t)\frac{\partial u}{\partial x}\frac{\partial\left(v\psi(x)\right)}{\partial x}+b_0(x,t)u\right]dxdt+$$

$$+\int_{Q_\tau}\left[c_2(x,t)\frac{\partial^2 u}{\partial x^2}+c_1(x,t)\frac{\partial u}{\partial x}+g(x,t,\frac{\partial u}{\partial t})-f(x,t)\right]v\psi(x)dxdt+$$

$$+\int_0^{+\infty}\left[\frac{\partial u}{\partial t}(x,\tau)v(x,\tau)-u_1(x)v(x,0)\right]\psi(x)dx=0 \qquad (19)$$

for any $\tau\in(0,T]$ and for any function $v\in L^2((0,T);H_0^{2,\psi}(0,+\infty))\cap L^p((0,T);L^{p,\psi}(0,+\infty))$ such that $\frac{\partial v}{\partial t}\in L^2((0,T);L^{2,\psi}(0,+\infty))$.

The main result. Let the conditions (Ψ), **(A), (B), (C), (G), (F), (U)** be satisfied. Then there exists a unique generalized solution u of the problem (15)-(18) in the domain Q_T, for which

$$\int_0^{+\infty}\left[\left(\frac{\partial u(x,\tau)}{\partial t}\right)^2+\left(\frac{\partial^2 u(x,\tau)}{\partial x^2}\right)^2\right]\psi(x)dx+\int_{Q_\tau}\left[\left(\frac{\partial^3 u}{\partial t\partial x^2}\right)^2+\left|\frac{\partial u}{\partial t}\right|^p\right]\psi(x)dxdt\leq$$

$$\leq C_0\left(\|f\|^{p'}_{L^{p'}\left((0,T);L^{p',\psi}(0;+\infty)\right)}+\|u_0\|^2_{H_0^{1,\psi}(0;+\infty)}+\|u_1\|^2_{L^{2,\psi}(0;+\infty)}\right) \qquad (20)$$

for any $\tau\in(0,T]$, where C_0 is a positive constant that depends on p, ψ and the coefficients a_2, a_1, a_0, b_2, b_1, b_0, c_2, c_1, g of equation (15).

Methodology for obtaining the result. *Existence*. We choose a sequence of intervals $\{(0,k)\}$, $k=2,3,\ldots$ Let us denote by $f^k(x,t)=\begin{cases}f(x,t), & x\leq k,\\ 0, & x>k,\end{cases}$

$u_0^k(x)=u_0(x)\xi_k(x)$, where $\xi_k\in C^2(\mathbf{R})$, $\xi_k(x)=\begin{cases}1, & x\leq k-1,\\ 0, & x>k,\end{cases}$ $0\leq\xi_k(x)\leq 1$, $u_1^k(x)=\begin{cases}u_1(x), & x\leq k,\\ 0, & x>k.\end{cases}$ It is obvious that $u_0^k\in H_0^2((0,k))$, $u_0^k\to u_0$ is strongly in

$H_0^{2,\psi}(0,+\infty)$, $u_1^k \to u_1$ is strongly in $L^{2,\psi}(0,+\infty)$, $k \to \infty$. We consider in Q_T^k ($k = 2,3,...$) a mixed problem:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(a_2(x,t)\frac{\partial^3 u}{\partial x^2 \partial t}\right) - \frac{\partial}{\partial x}\left(a_1(x,t)\frac{\partial u}{\partial x \partial t}\right) +$$

$$+a_0(x,t)\frac{\partial u}{\partial t} + \frac{\partial}{\partial x^2}\left(b_2(x,t)\frac{\partial^2 u}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(b_1(x,t)\frac{\partial u}{\partial x}\right) + b_0(x,t)u +$$

$$+c_2(x,t)\frac{\partial^2 u}{\partial x^2} + c_1(x,t)\frac{\partial u}{\partial x} + g(x,t,\frac{\partial u}{\partial t}) = f^k(x,t), \quad (21)$$

$$u(x,0) = u_0^k(x), \quad (22)$$

$$\frac{\partial u(x,0)}{\partial t} = u_1^k(x), \quad (23)$$

$$u(0,t) = \frac{\partial^2 u(0,t)}{\partial x^2} = 0, \quad u(k,t) = \frac{\partial^2 u(k,t)}{\partial x^2} = 0. \quad (24)$$

Based on the results of [6], we can assert that the generalized solution u^k of the problem (21)-(24) in Q_T^k exists and unique. Let us consider the sequence of problems of the form (21)-(24) for $k=2$, $k=3$,..., setting u^k zero at $Q_T \setminus Q_T^k$. We obtain a sequence of solutions to problems (15)-(18) in Q_T, which for convenience we denote again by $\{u^k\}$. We assume $v = \frac{\partial u^k}{\partial t}$ and $f = f^k$ in the integral equality (19). Using the conditions of the theorem and considerations similar to [7, p. 21], we conclude from equation (15), in particular, that $\frac{\partial^2 u^k}{\partial t^2} \in L^2\left((0,T);\left(H_0^{2,\psi}(0,+\infty) \cap L^{p,\psi}(0,+\infty)\right)^*\right)$ for any $k > 1$, so that $\int_0^\tau \left\langle \frac{\partial^2 u^k}{\partial t^2}, \frac{\partial u^k}{\partial t} \right\rangle \psi(x)dt$ there exists for any $\tau \in (0,T]$. Moreover, the integration by parts of $\int_0^\tau \left\langle \frac{\partial^2 u^k}{\partial t^2}, \frac{\partial u^k}{\partial t} \right\rangle \psi(x)dt$ can be used. Therefore, we obtain

$$\frac{1}{2}\int_0^{+\infty}\left(\frac{\partial u^k(x,\tau)}{\partial t}\right)^2\psi(x)dx+\int_{Q_\tau}a_2(x,t)\frac{\partial^3 u^k}{\partial t\partial x^2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u^k}{\partial t}\psi(x)\right)dxdt+$$

$$+\int_{Q_\tau}\left[a_1(x,t)\frac{\partial^2 u}{\partial t\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u^k}{\partial t}\psi(x)\right)+a_0(x,t)\left(\frac{\partial u^k}{\partial t}\right)^2\psi(x)\right]dxdt+$$

$$+\int_{Q_\tau}\left[b_2(x,t)\frac{\partial^2 u^k}{\partial x^2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u^k}{\partial t}\psi(x)\right)+b_1(x,t)\frac{\partial u^k}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u^k}{\partial t}\psi(x)\right)\right]dxdt+$$

$$+\int_{Q_\tau}\left[b_0(x,t)u^k+c_2(x,t)\frac{\partial^2 u^k}{\partial x^2}+c_1(x,t)\frac{\partial u^k}{\partial x}\right]\frac{\partial u^k}{\partial t}\psi(x)dxdt+$$

$$+\int_{Q_\tau}\left[g(x,t,\frac{\partial u^k}{\partial t})-f(x,t)\right]\frac{\partial u^k}{\partial t}\psi(x)dxdt-\frac{1}{2}\int_0^{+\infty}\left(u_1^k(x)\right)^2\psi(x)dx=0 \qquad (25)$$

Taking into account the conditions of the theorem, let us transform and estimate the integrals of equality (25):

$$\int_{Q_\tau}a_2(x,t)\frac{\partial^3 u^k}{\partial t\partial x^2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u^k}{\partial t}\psi(x)\right)dxdt=\int_{Q_\tau}a_2(x,t)\left(\frac{\partial^3 u^k}{\partial t\partial x^2}\right)^2\psi(x)dxdt+$$

$$+2\int_{Q_\tau}a_2(x,t)\frac{\partial^3 u^k}{\partial t\partial x^2}\frac{\partial^2 u^k}{\partial t\partial x}\psi'(x)+a_2(x,t)\frac{\partial^3 u^k}{\partial t\partial x^2}\frac{\partial u^k}{\partial t}\psi''(x)dxdt=I_1+I_2+I_3.$$

We estimate the integral I_1 as follows: $I_1\geq a_{0,2}\int_{Q_\tau}\left(\frac{\partial^3 u^k}{\partial t\partial x^2}\right)^2\psi(x)dxdt.$

We use the inequality to estimate the integral I_2

$$\int_0^{+\infty}\left(\frac{\partial w(x,\tau)}{\partial x}\right)^2\psi(x)dx\leq\delta_0\int_0^{+\infty}\left(\frac{\partial^2 w(x,\tau)}{\partial x^2}\right)^2\psi(x)dx+C(\delta_0)\int_0^{+\infty}w^2\psi(x)dx, \qquad (26)$$

$\delta_0>0$, $C(\delta_0)>0$, which is satisfied for any function $w\in H_0^{2,\psi}(0,+\infty)$. Let us prove inequality (26), by writing the obvious equality

$$\int_0^{+\infty}\frac{\partial^2 w}{\partial x^2}w\psi(x)dx=\int_0^{+\infty}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}w\psi(x)\right)dx-\int_0^{+\infty}\left(\frac{\partial w}{\partial x}\right)^2\psi(x)dx-$$

 ISBN 978-3-98924-038-4

$$-\int_0^{+\infty}\frac{\partial w}{\partial x}w\psi'(x)dx=-\int_0^{+\infty}\left(\frac{\partial w}{\partial x}\right)^2\psi(x)dx-\int_0^{+\infty}\frac{\partial w}{\partial x}w\psi'(x)dx$$

.

Then, by Cauchy's inequality, we obtain

$$\int_0^{+\infty}\left(\frac{\partial w(x,\tau)}{\partial x}\right)^2\psi(x)dx=-\int_0^{+\infty}\frac{\partial^2 w}{\partial x^2}w\psi(x)dx-\int_0^{+\infty}\frac{\partial w}{\partial x}w\psi'(x)dx\leq$$

$$\leq\frac{\delta}{2}\int_0^{+\infty}\left(\frac{\partial^2 w}{\partial x^2}\right)^2\psi(x)dx+\frac{1}{2\delta}\int_0^{+\infty}w^2\psi(x)dx+\frac{a}{2}\int_0^{+\infty}\left(\frac{\partial w}{\partial x}\right)^2\psi(x)dx+M(a)\int_0^{+\infty}w^2\psi(x)dx,$$

where $\delta>0$, $a\leq 1$, $M(a)>0$. It is easy to derive (26) from this inequality.

Applying (26), we obtain

$$I_2\leq C_1\delta_1\int_{Q_\tau}\left(\frac{\partial^3 u^k}{\partial t\partial x^2}\right)^2\psi(x)dxdt+C_2\int_{Q_\tau}\left(\frac{\partial u^k}{\partial t}\right)^2\psi(x)dxdt.$$

Furthermore,

$$I_3\leq C_3\delta_2\int_{Q_\tau}\left(\frac{\partial^3 u^k}{\partial t\partial x^2}\right)^2\psi(x)dxdt+C_4\int_{Q_\tau}\left(\frac{\partial u^k}{\partial t}\right)^2\psi(x)dxdt.$$

In the last two estimates δ_1, δ_2 are arbitrary positive constants, the constant $C_1>0$ depends on $a_2=\underset{(x,t)\in Q_T}{\operatorname{essup}}|a_2(x,t)|$, M_1, the constant $C_2>0$ depends on a_2, M_1, δ_1, the constant $C_3>0$ depends on a_2, M_2, the constant $C_4>0$ depends on a_2, δ_2, M_2.

Let us transform it

$$\int_{Q_\tau}b_2(x,t)\frac{\partial^2 u^k}{\partial x^2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u^k}{\partial t}\psi(x)\right)dxdt=\int_{Q_\tau}b_2(x,t)\frac{\partial^2 u^k}{\partial x^2}\frac{\partial^3 u^k}{\partial t\partial x^2}\psi(x)dxdt+$$

$$+2\int_{Q_\tau}b_2(x,t)\frac{\partial^2 u^k}{\partial x^2}\frac{\partial^2 u^k}{\partial t\partial x}\psi'(x)dxdt+\int_{Q_\tau}b_2(x,t)\frac{\partial^2 u^k}{\partial x^2}\frac{\partial u^k}{\partial t}\psi''(x)dxdt=I_4+I_5+I_6.$$

We estimate the integrals I_4, I_5, I_6:

$$I_4=\frac{1}{2}\int_{Q_\tau}\frac{\partial}{\partial t}\left(b_2(x,t)\left(\frac{\partial^2 u^k}{\partial x^2}\right)^2\psi(x)\right)dxdt-\frac{1}{2}\int_{Q_\tau}\frac{\partial b_2(x,t)}{\partial t}\times$$

$$\times\left(\frac{\partial^2 u^k}{\partial x^2}\right)^2\psi(x)dxdt\geq\frac{b_{0,2}}{2}\int_0^{+\infty}\left(\frac{\partial^2 u^k(x,\tau)}{\partial^2 x}\right)^2\psi(x)dx-$$

$$-\frac{1}{2}\int_0^{+\infty} b_2(x,0)\left(\frac{\partial^2 u_0^k}{\partial^2 x}\right)^2 \psi(x)dx - C_5\int_{Q_\tau}\left(\frac{\partial^2 u^k}{\partial^2 x}\right)^2 \psi(x)dxdt,$$

C_5 is some positive constant that depends on $\underset{x\in Q_T}{\text{essup}}\left|\frac{\partial b_2(x,t)}{\partial t}\right|$;

$$I_5 \le C_6\delta_3\int_{Q_\tau}\left(\frac{\partial^3 u^k}{\partial t\partial x^2}\right)^2 \psi(x)dxdt + C_7\int_{Q_\tau}\left(\frac{\partial^2 u^k}{\partial x^2}\right)^2 \psi(x)dxdt + C_8\int_{Q_\tau}\left(\frac{\partial u^k}{\partial t}\right)^2 \psi(x)dxdt,$$

δ_3 is an arbitrary positive constant, C_6, C_7, C_8 are some positive constants depending on $b_2 = \underset{x\in Q_T}{\text{essup}}|b_2(x,t)|$, M_1;

$$I_6 \le C_9\int_{Q_\tau}\left(\frac{\partial^2 u^k}{\partial x^2}\right)^2 \psi(x)dxdt + C_{10}\int_{Q_\tau}\left(\frac{\partial^2 u^k}{\partial x^2}\right)^2 \psi(x)dxdt,$$

C_9 is some positive constant depending on b_2, M_2; C_{10} is some positive constant depending on b_2, M_2. Moreover,

$$\int_{Q_\tau} c_2(x,t)\frac{\partial^2 u^k}{\partial x^2}\frac{\partial u^k}{\partial t}\psi(x)dxdt \le \frac{c_2}{2}\int_{Q_\tau}\left(\frac{\partial^2 u^k}{\partial x^2}\right)^2 \psi(x)dxdt + \frac{c_2}{2}\int_{Q_\tau}\left(\frac{\partial u^k}{\partial t}\right)^2 \psi(x)dxdt,$$

where $c_2 = \underset{x\in Q_T}{\text{essup}}|c_2(x,t)|$. Analogously to the above transformations and estimates, we obtain

$$\int_{Q_\tau} a_1(x,t)\frac{\partial^2 u^k}{\partial t\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u^k}{\partial t}\psi(x)\right)dxdt = \int_{Q_\tau} a_1(x,t)\left(\frac{\partial^2 u^k}{\partial t\partial x}\right)^2 \psi(x)dxdt +$$

$$+\int_{Q_\tau} a_1(x,t)\frac{\partial^2 u^k}{\partial t\partial x}\frac{\partial u^k}{\partial t}\psi'(x)dxdt = I_7 + I_8;$$

$$I_7 \le C_{11}\delta_4\int_{Q_\tau}\left(\frac{\partial^3 u^k}{\partial t\partial x^2}\right)^2 \psi(x)dxdt + C_{12}\int_{Q_\tau}\left(\frac{\partial u^k}{\partial t}\right)^2 \psi(x)dxdt,$$

$$I_8 \le C_{13}\delta_5\int_{Q_\tau}\left(\frac{\partial^3 u^k}{\partial t\partial x^2}\right)^2 \psi(x)dxdt + C_{14}\int_{Q_\tau}\left(\frac{\partial u^k}{\partial t}\right)^2 \psi(x)dxdt,$$

 ISBN 978-3-98924-038-4

δ_4, δ_5 are arbitrary positive constants, $C_{11}>0$ depends on $a_1=\underset{(x,t)\in Q_T}{\operatorname{essup}}|a_1(x,t)|$, $C_{13}>0$ depends on a_1, ψ, $C_{12}>0$ depends on a_1, δ_4, ψ, $C_{14}>0$ depends on a_1, δ_5, M_1. Furthermore, we derive

$$\int_{Q_\tau}\left[b_1(x,t)\frac{\partial u^k}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u^k}{\partial t}\psi(x)\right)\right]dxdt=\int_{Q_\tau}b_1(x,t)\frac{\partial u^k}{\partial x}\frac{\partial^2 u^k}{\partial x\partial t}\psi(x)dxdt+$$

$$+\int_{Q_\tau}b_1(x,t)\frac{\partial u^k}{\partial x}\frac{\partial u^k}{\partial t}\psi'(x)dxdt=I_9+I_{10};$$

$$I_9=\int_{Q_\tau}b_1(x,t)\frac{\partial^2 u^k}{\partial t\partial x}\frac{\partial}{\partial x}\left[u_0^k(x)+\int_0^t\frac{\partial u^k(x,\tau)}{\partial t}d\tau\right]\psi(x)dxdt\le$$

$$\le C_{15}(b_1,T)\delta_6\int_{Q_\tau}\left(\frac{\partial^3 u^k}{\partial t\partial x^2}\right)^2\psi(x)dxdt+C_{16}(b_1,T,\delta_6)\times$$

$$\times\int_{Q_\tau}\left(\frac{\partial u^k}{\partial t}\right)^2\psi(x)dxdt+C_{17}(b_1,\delta_6)\int_0^{+\infty}\left(\frac{\partial^2 u_0^k}{\partial x^2}\right)^2\psi(x)dx,$$

$b_1=\underset{(x,t)\in Q_T}{\operatorname{essup}}|b_1(x,t)|$, $\delta_6>0$ – arbitrary constant, $C_{15}>0$, $C_{16}>0$, $C_{17}>0$;

$$I_{10}=\int_{Q_\tau}b_1(x,t)\frac{\partial u^k}{\partial t}\frac{\partial}{\partial x}\left[u_0^k(x)+\int_0^t\frac{\partial u^k(x,\tau)}{\partial t}d\tau\right]\psi'(x)dxdt\le$$

$$\le C_{18}(b_1,M_1,T)\delta_7\int_{Q_\tau}\left(\frac{\partial^3 u^k}{\partial t\partial x^2}\right)^2\psi(x)dxdt+C_{19}(b_1,M_1,\delta_7,T)\times$$

$$\times\int_{Q_\tau}\left(\frac{\partial u^k}{\partial t}\right)^2\psi(x)dx+C_{20}(b_1,M_1,\delta_7)\int_0^{+\infty}\left(\frac{\partial u_0^k}{\partial x}\right)^2\psi(x)dx,$$

δ_7 – arbitrary positive constant, $C_{18}>0$, $C_{19}>0$, $C_{20}>0$.

We complete the estimation of the equality integrals (25):

$$\int_{Q_\tau}a_0(x,t)\left(\frac{\partial u^k}{\partial t}\right)^2\psi(x)dxdt\le a_0\int_{Q_\tau}\left(\frac{\partial u^k}{\partial t}\right)^2\psi(x)dxdt,$$

$$\int\limits_{Q_\tau} b_0(x,t)u^k \frac{\partial u^k}{\partial t}\psi(x)dxdt = \int\limits_{Q_\tau} b_0(x,t)u^k \frac{\partial u^k}{\partial t}\left[u_0^k(x) + \int\limits_0^t \frac{\partial u^k(x,\tau)}{\partial t}d\tau\right]\times$$

$$\times\psi(x)dxdt \le b_0\left(T+\frac{1}{2}\right)\int\limits_{Q_\tau}\left(\frac{\partial u^k}{\partial t}\right)^2\psi(x)dxdt + \frac{b_0 T}{2}\int\limits_0^{+\infty}\left(u_0^k\right)^2\psi(x)dx,$$

$$a_0 = \operatorname*{essup}_{(x,t)\in Q_T}\left|a_0(x,t)\right|, \quad b_0 = \operatorname*{essup}_{(x,t)\in Q_T}\left|b_0(x,t)\right|;$$

$$\int\limits_{Q_\tau} c_1(x,t)\frac{\partial u^k}{\partial x}\frac{\partial u^k}{\partial t}\psi(x)dxdt = \int\limits_{Q_\tau} c_1(x,t)\frac{\partial u^k}{\partial t}\frac{\partial}{\partial x}\left[u_0^k(x) + \right.$$

$$\left.+\int\limits_0^t \frac{\partial u^k}{\partial t}(x,\tau)d\tau\right]\psi(x)dxdt \le C_{21}(c_1,T)\delta_8\int\limits_{Q_\tau}\left(\frac{\partial^3 u^k}{\partial t\partial x^2}\right)^2\psi(x)dxdt +$$

$$+C_{22}(c_1,\delta_8,T)\int\limits_{Q_\tau}\left(\frac{\partial u^k}{\partial t}\right)^2\psi(x)dx + C_{23}(c_1,\delta_8)\int\limits_0^{+\infty}\left(\frac{\partial^2 u_0^k}{\partial x^2}\right)^2\psi(x)dx,$$

δ_8 – arbitrary positive constant, $c_1 = \operatorname*{essup}_{(x,t)\in Q_T}\left|c_1(x,t)\right|$, $C_{21} > 0$, $C_{22} > 0$, $C_{23} > 0$;

$$\int\limits_{Q_\tau} g\left(x,t,\frac{\partial u^k}{\partial t}\right)\frac{\partial u^k}{\partial t}\psi(x)dxdt \ge g_0\int\limits_{Q_\tau}\left|\frac{\partial u^k}{\partial t}\right|^p\psi(x)dxdt;$$

$$\int\limits_{Q_\tau} f^k\frac{\partial u^k}{\partial t}\psi(x)dxdt \le \delta_9\int\limits_{Q_\tau}\left|\frac{\partial u^k}{\partial t}\right|^p\psi(x)dxdt + C_{24}\int\limits_{Q_\tau}\left|f^k\right|^{p'}\psi(x)dxdt,$$

δ_9 – arbitrary positive constant, C_{24} – some positive constant that depends on δ_9, p.

Taking into account the above estimates, we obtain from equality (25):

$$\int\limits_0^{+\infty}\left(\frac{\partial u^k(x,\tau)}{\partial t}\right)^2\psi(x)dx + 2\left(a_{0,2} - C_1\delta_1 - C_3\delta_2 - C_6\delta_3 - C_{11}\delta_4 - C_{13}\delta_5 - C_{15}\delta_6 - C_{18}\delta_7 - \right.$$

$$\left.-C_{21}\delta_8\right)\int\limits_{Q_\tau}\left(\frac{\partial^3 u^k}{\partial t\partial x^2}\right)^2\psi(x)dxdt + b_{0,2}\int\limits_0^{+\infty}\left(\frac{\partial^2 u^k(x,\tau)}{\partial x^2}\right)^2\psi(x)dx +$$

$$+2\left(g_0 - \delta_9\right)\int\limits_{Q_\tau}\left|\frac{u^k}{\partial t}\right|^p\psi(x)dxdt \le 2\left(C_2 + C_4 + C_8 + C_9 + \frac{c_2}{2} + \right.$$

 ISBN 978-3-98924-038-4

$$+C_{12}+C_{14}+C_{16}+C_{19}+a_0+b_0\left(T+\frac{1}{2}\right)+C_{22}\Bigg)\int_{Q_\tau}\left(\frac{\partial u^k}{\partial t}\right)^2\psi(x)dxdt+$$

$$+2\left(C_5+C_7+C_{10}+\frac{c_2}{2}\right)\int_{Q_\tau}\left(\frac{\partial^2 u^k}{\partial x^2}\right)^2 dxdt+2C_{24}\int_{Q_\tau}\left|f^k\right|^{p'}\psi(x)dxdt+$$

$$+2(C_{17}+C_{20}+C_{23})\int_0^{+\infty}\left(\frac{\partial u_0^k}{\partial x}\right)^2\psi(x)dx+b_0T\int_0^{+\infty}\left(u_0^k\right)^2\psi(x)dx+$$

$$+\int_0^{+\infty}\left(u_1^k\right)^2\psi(x)dx+\int_0^{+\infty}b_2(x,0)\frac{\partial^2 u_0^k(x)}{\partial x^2}\psi(x)dx.$$

Letting us denote $y(\tau)=\int_0^{+\infty}\left[\left(\frac{\partial u^k(x,\tau)}{\partial t}\right)^2+\left(\frac{\partial^2 u^k(x,\tau)}{\partial x^2}\right)^2\right]\psi(x)dx$, we obtain an estimate of $y(\tau)\le C_{25}+C_{26}\int_0^{\tau}y(t)dt$, where C_{25}, C_{26} are some positive constants independent of u^k. We then conclude, based on Gronwall's lemma, by choosing sufficiently small constants $\delta_1-\delta_9$, that

$$\int_0^{+\infty}\left[\left(\frac{u^k(x,\tau)}{\partial t}\right)^2+\left(\frac{\partial^2 u^k(x,\tau)}{\partial x^2}\right)^2\right]\psi(x)dx+$$

$$+\int_{Q_\tau}\left[\left(\frac{\partial u^k}{\partial t}\right)^2+\left(\frac{\partial^3 u^k(x,\tau)}{\partial t\partial x^2}\right)^2\right]\psi(x)dxdt\le C_{27} \tag{27}$$

for all $0<\tau\le T$, C_{27} is a positive constant independent of k. From inequality (27) follows the existence of a subsequence $\{u^{k_s}\}\subset\{u^k\}$ such that $u^{k_s}\to u$ * - weakly in $L^\infty((0,T);H_0^{2,\psi}(0,+\infty))$, $\frac{\partial u^k}{\partial t}\to\frac{\partial u}{\partial t}$ * weakly in $L^\infty((0,T);L^{2,\psi}(0,+\infty))$, $\frac{\partial u^k}{\partial t}\to\frac{\partial u}{\partial t}$ weakly in $L^2((0,T);H_0^{2,\psi}(0,+\infty))\cap L^p((0,T);L^{p,\psi}(0,+\infty))$ at $k_s\to\infty$. Applying [7, p.20, Lemma 1.2], we obtain $u\in C\left([0,T];H_0^2(0,+\infty)\right)$. Similarly to [7, p. 234], we also obtain that $\frac{\partial u}{\partial t} \in C\left([0,T];(H_0^{2,\psi}(0,+\infty) \cap L^{p,\psi}(0,+\infty))^*\right)$. Since it $u^{k_s}(\cdot,0) \to u(\cdot,0)$ is weak in $H_0^{2,\psi}(0,+\infty)$, $u_0^{k_s} \to u_0$ strong in $H_0^{2,\psi}(0,+\infty)$, then $u(x,0) = u_0(x)$, $x \in (0,+\infty)$. Similarly to [7, p. 236], we also show that $\frac{\partial u(x,0)}{\partial t} = u_1(x)$, $x \in (0,+\infty)$. Note that for any $k \in \mathbf{N}$

$$\left\| \frac{\partial u^k}{\partial t} \right\|_{L^p((0,T);L^{p,\psi}(0,+\infty))} \le C_{28}, \quad C_{28} = const > 0. \tag{28}$$

From inequality (28), taking into account the condition (**G**), it is easy to obtain

$$\int_{Q_T} \left| g\left(x,t,\frac{\partial u^k}{\partial t}\right) \right|^{p'} dxdt \le C_{29}, \quad C_{29} > 0. \tag{29}$$

We conclude from inequalities (28)-(29) (moving to subsequences if necessary) that $g\left(x,t,\frac{\partial u^k}{\partial t}\right) \to z$ is weak in $L^{p'}((0;T);L^{p',\psi}(0,+\infty))$. Similarly to [7, pp. 236-237], we show that $z = g\left(x,t,\frac{\partial u}{\partial t}\right)$. It u satisfies the integral identity (19), and conditions (16), (18). Thus, u is a generalized solution to the problem (15)-(18) in Q_T. The correctness of the estimate (20) is proved similarly to the derivation of inequality (27).

Uniqueness. If u^1 and u^2 are two arbitrary solutions of the problem (15) - (18), then similarly to (20), we can obtain an estimate

$$\int_0^{+\infty} \left[\left(\frac{\partial u^1(x,\tau)}{\partial t} - \frac{\partial u^2(x,\tau)}{\partial t} \right)^2 + \left(\frac{\partial^2 u^1(x,\tau)}{\partial x^2} - \frac{\partial^2 u^2(x,\tau)}{\partial x^2} \right)^2 \right] \psi(x)dx +$$

$$+\int_{Q_\tau} \left[\left(\frac{\partial^3 u^1}{\partial t \partial x^2} - \frac{\partial^3 u^2}{\partial t \partial x^2} \right)^2 + \left| \frac{\partial u^1}{\partial t} - \frac{\partial u^2}{\partial t} \right|^p \right] \psi(x)dxdt \le 0$$

for all $0 < \tau \le T$. Thus, $u^1 = u^2$ almost everywhere in Q_T.

Summary and conclusions

For most applied problems of nonlinear vibration theory, the effect of generalized internal dissipation forces in an oscillatory system is obvious. In particular, bending waves in rods are described by a fifth-order linear equation that takes into account the influence of dissipative forces on the dynamic process. The conditions of correctness (sufficient conditions of existence and uniqueness) of the solution in the mathematical model of vibrations under the action of nonlinear dissipative forces in terms of the Voigt-Kelvin theory are obtained. Correctness classes are obtained in the mathematical model of oscillations of bounded and unbounded bodies. The basis for further investigations of the existence of a unique generalized solution for the case of the mathematical model of vibration of unbounded rods and beams is provided.

The publication was performed within the framework of the state budget theme of the Ministry of Education and Science of Ukraine No. 0123U101691.
Recommended for publication by the Academic Council of the Institute of Applied Mathematics and Fundamental Sciences of Lviv Polytechnic National University, Protocol No. 68 of 20.02.2024.