KAPITEL 8 / *CHAPTER 8* [8]
THREE-DIMENSIONAL VIBRATION ANALYSIS OF LAMINATED SHELLS

DOI: 10.30890/2709-2313.2024-35-00-001

Introduction

Numerical analysis of the characteristics of beams, plates and shells of revolution in a static or dynamic state, resting on an elastic foundation, is usually based on approximate models of the elastic foundation [1]. The reaction of the foundation is described by differential operators acting on the deflections of elastic bodies. A large number of studies are devoted to the analysis of the influence of an elastic foundation on the linear or nonlinear vibrations of circular cylindrical shells [2]. In particular, natural frequencies of oscillations were obtained for simply supported cylindrical shells [3, 4]. Numerical values of characteristic coefficients of natural oscillations vary in a wide range for frequencies corresponding to radial, longitudinal and torsional modes [5, 6].

The three-dimensional case of free vibrations of thick-walled cylindrical shells immersed in a two-parameter elastic medium can also be characterized by a limited number of modes with different boundary conditions and with different combinations of characteristic coefficients. It should be noted that such properties of the elastic foundation as inertia also affect the natural vibrations of three-layer shells. In particular, the presence of an elastic medium significantly increases the frequencies of radial vibrations of three-layer shells with a thick filler [7, 8]. The numerical values of natural frequencies, as well as the form factors of vibrations, nonlinearly depend on the variable thickness of cylindrical isotropic and orthotropic shells. Experiments indicate an increase in the influence of an elastic foundation with an increase in the ratio of the maximum thickness to the minimum.

Local gradients of mechanical stresses on the surfaces of the functionally graded shell of reinforced composites, split into several layers and immersed in an elastic

[8] ***Authors:*** *Pysarenko Alexander Mykolayovich*
Number of characters: 15686
Author's sheets: *0,39*

foundation of the Winkler type, lead to a decrease in the frequencies of free oscillations. Analysis of the spectrum of nonlinear coefficients of the elastic foundation allows localizing the regions of increasing oscillation frequencies on the surface of cylindrical shells of laminar composites [9].

In the case of uniaxial tension, the effect of discretization of the graded layer into a number of homogeneous sublayers occurs, each of which is characterized by its own displacement coefficient. The increase in the rigidity of the elastic medium is the reason that the influence of geometric nonlinearity, material heterogeneity, the number of winding layers and the magnitude of the reinforcement angles on the oscillation frequencies is reduced.

Most of these methods were first applied to isotropic cylindrical shells and then extended to study the dynamic behavior of anisotropic and layered composite shells. However, despite the various methods of analytical and computational analysis of cylindrical shell structures, finding reliable and efficient approaches for the considered structures with different boundary conditions still remains a big problem.

Therefore, the aim of this paper is to introduce the Haar wavelet approach for the analysis of free vibrations of composite layered cylindrical shells. The free oscillation model used the Haar wavelet, which consists of pairs of piecewise constant functions and one of the simplest orthonormal wavelets with a compact support. A limited set of orthonormal wavelets generated from the same parent wavelet form a basis. The elements of the wavelet basis are orthonormal to each other and normalized to unit length. This property allows each wavelet coefficient to be calculated independently of other wavelets.

8.1. Haar wavelet discretization method

The Haar wavelet family $h_i(\xi)$, defined for $\xi \in [0, 1]$. The matrix H of Haar characteristic coefficients based on l is defined as $H(i, l) = h_i(\xi_i)$. The corresponding matrix $S^{(\alpha)}(i, l)$ of integral transformations has dimensions of $2K \times 2K$. Let us consider a model of a composite layered cylindrical shell. In this model, the length, average radius and thickness of the shell are designated as *L*, *R* and *a*, respectively. The main surface of the shell can be considered as the median surface on which the orthogonal coordinate system (x, θ and z) is fixed. The x, θ and z axes are taken in the axial, circumferential and radial directions, respectively. The displacements of the shell in the x, θ and z directions are designated as b, c and u.

The deformation at the mean surface (δ_0) and the change in curvature (μ_0) during deformation with transposition operator *T* are functions of the displacement

$$\delta_0 = [\delta_1, \delta_2, \delta_3]^T, \qquad \mu_0 = [\mu_1, \mu_2 \mu_3]^T \tag{1}$$

$$\delta_1 = \partial b / \partial x, \quad \delta_2 = \partial c / R\partial\theta, \quad \delta_3 = \partial c / \partial x \tag{2}$$

$$\mu_1 = -\partial^2 u / \partial x^2, \quad \mu_2 = -\partial c / R^2 \partial\theta, \quad \mu_3 = -\partial^2 u / R\partial x \partial\theta . \tag{3}$$

The governing equations for vibrations can be expressed as in the following form of stiffness matrixes *Q*, *N* and differential operators $M_{ij} = M_{ij}(Q, N, x, \theta)$

$$L_{11}b + L_{12}c + L_{13}u = (Q, N)\frac{\partial^2 b}{\partial t^2}, \tag{4}$$

$$L_{21}b + L_{22}c + L_{23}u = (Q, N)\frac{\partial^2 c}{\partial t^2}, \tag{5}$$

$$L_{31}b + L_{32}c + L_{33}u = (Q, N)\frac{\partial^2 u}{\partial t^2}. \tag{6}$$

Thus the displacement components b, c, u are

$$b(x,\theta,t) = B(x)\cos(m\theta)\exp(i\omega t), \tag{7}$$

$$c(x,\theta,t) = C(x)\sin(m\theta)\exp(i\omega t), \tag{8}$$

$$u(x,\theta,t) = U(x)\cos(m\theta)\exp(i\omega t), \tag{9}$$

where ω is the angular frequency of vibration and $m = n$ is the circumferential wave numbers, t is the time variable.

This model considers boundary conditions of the following types: *EC1* (clamped edge), *EC2* (simply supported edge) and *EC3* (free edge). They are defined as follows:

 ISBN 978-3-98924-070-4

$$EC1: B = 0,\ C = 0,\ U = 0,\ \frac{dU}{dx} = 0 , \quad (10)$$

$$EC2: C = 0, U = 0, Nx = 0,\ Mx = 0 , \quad (11)$$

$$EC3: M_x = N_x + \frac{M_{x0}}{R},\ M_x = Q_x . \quad (12)$$

The force N and moment M resultants can be described based on the following relationships

$$\overline{N} = [N_x, N_\theta, N_{x\theta}]^T , \qquad \overline{M} = [M_x, M_\theta, M_{x\theta}]^T , \quad (13)$$

$$(N_x, N_\theta, N_{x\theta}) = \int_Z (\sigma_{xx}, \sigma_{\theta\theta}, \tau_{x\theta}) dz , \quad (14)$$

$$(M_x, M_\theta, M_{x\theta}) = \int_Z (\sigma_{xx}, \sigma_{\theta\theta}, \tau_{x\theta}) zdz . \quad (15)$$

Using the boundary conditions, additional equations can be obtained. The calculation method offers an exact solution for cylindrical shells with various boundary conditions. All classical boundary conditions can be easily achieved. In particular, the clamped-clamped boundary condition will be given as a representative solution procedure. The discretized forms of considered boundary condition can be written as

$$B(0) = \sum_{i=1}^{m} q_i s_{1i}(0), \quad B(1) = \sum_{i=1}^{m} q_i s_{1i}(1) + \frac{dB(0)}{d\xi}, \quad (16)$$

$$C(0) = \sum_{i=1}^{m} n_i s_{2i}(0), \quad C(1) = \sum_{i=1}^{m} n_i s_{2i}(1) + \frac{dC(0)}{d\xi}, \quad (17)$$

$$U(0) = \sum_{i=1}^{m} r_i s_{3i}(0) , \quad (18)$$

$$U(1) = \sum_{i=1}^{m} r_i s_{3i}(1) + \frac{1}{6}\frac{d^3U(0)}{d\xi^3} + \frac{1}{2}\frac{d^2U(0)}{d\xi^2} + \frac{dU(0)}{d\xi} , \quad (19)$$

$$\frac{dU(0)}{d\xi} = \sum_{i=1}^{m} r_i s_{4i}(0) , \quad (20)$$

$$\frac{dU(1)}{d\xi} = \sum_{i=1}^{m} r_i s_{4i}(1) + \frac{1}{2}\frac{d^3U(0)}{d\xi^3} + \frac{d^2U(0)}{d\xi^2} + \frac{dU(0)}{d\xi} . \quad (21)$$

By solving Eq. (13 - 18), the boundary conditions can be achieved and summed up in matrix forms, respectively

$$B_e = \begin{bmatrix} B(0) \\ B(1) \end{bmatrix} = S_1 \begin{bmatrix} q \\ k \end{bmatrix}, \quad C_e = \begin{bmatrix} C(0) \\ C(1) \end{bmatrix} = S_2 \begin{bmatrix} n \\ f \end{bmatrix}, \quad (22)$$

$$U_e = \begin{bmatrix} U(0) & U(1) & \frac{dU(0)}{d\xi} & \frac{dU(1)}{d\xi} \end{bmatrix}^T = S_3 \begin{bmatrix} r \\ j \end{bmatrix}, \quad (23)$$

$$q = (q_1, q_2, ..., q_m)^T, \quad k = \left(\frac{dB(0)}{d\xi}, B(0) \right)^T, \quad (24)$$

$$n = (n_1, n_2, ..., n_m)^T, \quad f = \left(\frac{dC(0)}{d\xi}, C(0) \right)^T, \quad (25)$$

$$r = (r_1, r_2, ..., r_m)^T, \quad j = \left(\frac{dU(0)}{d\xi}, U(0) \right)^T, \quad (26)$$

$$S_1 = S_2 = \begin{bmatrix} s_{21}(\xi_1) & s_{22}(\xi_1) & \cdots & s_{2m}(\xi_1) & \xi_1 & 1 \\ s_{21}(\xi_2) & s_{22}(\xi_2) & \cdots & s_{2m}(\xi_2) & \xi_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ s_{21}(\xi_m) & s_{22}(\xi_m) & \cdots & s_{2m}(\xi_m) & \xi_m & 1 \end{bmatrix}, \quad (27)$$

$$S_3 = \begin{bmatrix} s_{31}(\xi_1) & s_{32}(\xi_1) & \cdots & s_{3m}(\xi_1) & \frac{\xi_1^3}{6} & \frac{\xi_1^2}{2} & \xi_1 & 1 \\ s_{31}(\xi_2) & s_{32}(\xi_2) & \cdots & s_{3m}(\xi_2) & \frac{\xi_2^3}{6} & \frac{\xi_2^2}{2} & \xi_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{31}(\xi_m) & s_{32}(\xi_m) & & s_{3m}(\xi_m) & \frac{\xi_m^3}{6} & \frac{\xi_m^2}{2} & \xi_m & 1 \end{bmatrix}, \quad (28)$$

 ISBN 978-3-98924-070-4

$$S_4 = S_5 = \begin{bmatrix} s_{21}(0) & s_{22}(0) & \cdots & s_{2m}(0) & 0 & 1 \\ s_{21}(1) & s_{22}(1) & \cdots & s_{2m}(1) & 1 & 1 \end{bmatrix}. \quad (29)$$

In this case, the following matrix equation can be written

$$\begin{bmatrix} B \\ B_e \end{bmatrix} = \begin{bmatrix} S_1 \\ S_3 \end{bmatrix} \begin{bmatrix} q \\ k \end{bmatrix} = \chi_1 \begin{bmatrix} q \\ k \end{bmatrix}, \quad (30)$$

$$\begin{bmatrix} C \\ C_e \end{bmatrix} = \begin{bmatrix} S_3 \\ S_4 \end{bmatrix} \begin{bmatrix} n \\ f \end{bmatrix} = \chi_2 \begin{bmatrix} n \\ f \end{bmatrix}, \quad (31)$$

$$\begin{bmatrix} U \\ U_e \end{bmatrix} = \begin{bmatrix} S_2 \\ S_5 \end{bmatrix} \begin{bmatrix} r \\ j \end{bmatrix} = \chi_3 \begin{bmatrix} r \\ j \end{bmatrix}. \quad (32)$$

In this case, the unknown coefficients χ_1, χ_2 and χ_3 can be determined as follows

$$\begin{bmatrix} q \\ k \end{bmatrix} = \chi_1^{-1} \begin{bmatrix} B \\ B_e \end{bmatrix}, \quad \begin{bmatrix} n \\ f \end{bmatrix} = \chi_2^{-1} \begin{bmatrix} C \\ C_e \end{bmatrix}, \quad \begin{bmatrix} r \\ j \end{bmatrix} = \chi_3^{-1} \begin{bmatrix} U \\ U_e \end{bmatrix}. \quad (33)$$

In the next step, all displacements and their derivatives can be expressed using Haar wavelet series and their integrals. The following characteristic form can be considered to describe the process of free oscillations

$$\frac{d^4U}{dx^4} = H_1 \chi_3^{-1} \begin{bmatrix} U \\ U_e \end{bmatrix} = B_1^* U + B_2^* U_e, \quad (34)$$

where B_1* and B_2* are the characteristic matrices for $H_1\chi_3^{-1}$.

The fourth-order derivative satisfies the following relation

$$\frac{d^4U}{dx^4} = \left(\frac{d^4U(\xi_1)}{dx^4}, \frac{d^4U(\xi_2)}{dx^4}, \ldots, \frac{d^4U(\xi_m)}{dx^4} \right)^T. \quad (35)$$

The Haar wavelet discretization method was used to discretize the derivatives in the control equations in terms of displacements and boundary conditions [10]. A necessary condition for solving the finite field problem is the transformation of the displacement field into a unit interval. Transformation of a series of wavelets leads to a discrete system of algebraic equations with respect to one normalized variable ξ. The higher-order derivatives of these solutions with respect to the axial coordinate can be expanded in terms of completed Haar wavelets.

Using boundary conditions, additional equations can be obtained. The current method offers an exact solution for cylindrical shells with different boundary conditions. All classical boundary conditions can be easily calculated. The governing equations and the corresponding boundary condition equations were discretized using wavelet transforms. From the above procedures, a general relationship was obtained for the displacement vector $X = [B, C, U]$, displacement matrix A, and local masses matrix W of laminated composites cylindrical shells

$$\left(A - \omega^2 W\right)X = 0 . \quad (36)$$

8.2. Results of the calculation model

The following values of physical quantities were used in the calculation part of the model: $R = 1.15$ m; $L/R = 4.3$; $a/\mathrm{R} = 0.02$; $E_2 = 14$ GPa; $E_1/E_2 = k$, $k \in [2.0 – 20]$; $G_{12} = 5.2$ GPa (shear modulus); $\rho = 1620$ kg/m^3. The following ratio was used as the reference frequency $\Omega = \omega R(\rho_0/E_2)^{0.5}$. The base frequency was calculated for both the three main boundary conditions *EC1*, *EC2* and *EC3* and for the intermediate boundary condition *EC1-EC2*.

An additional numerical analysis was performed to investigate the effect of complex lamination patterns on the frequencies of laminated cylindrical shells. The frequency parameters were determined for cross-laminated cylindrical shells. These shells had a small thickness ratio ($a/R = 0.02$) and a moderate length ($d' = L/R$). In addition, for simplicity, it was assumed that all layers had the same thickness.

The frequency parameters are classified not according to their wave number value, but according to their order in the direction of the larger radius of curvature.

The stiffness in the presence of a large number of shells can be maximum, and thus the frequency value is also the highest.

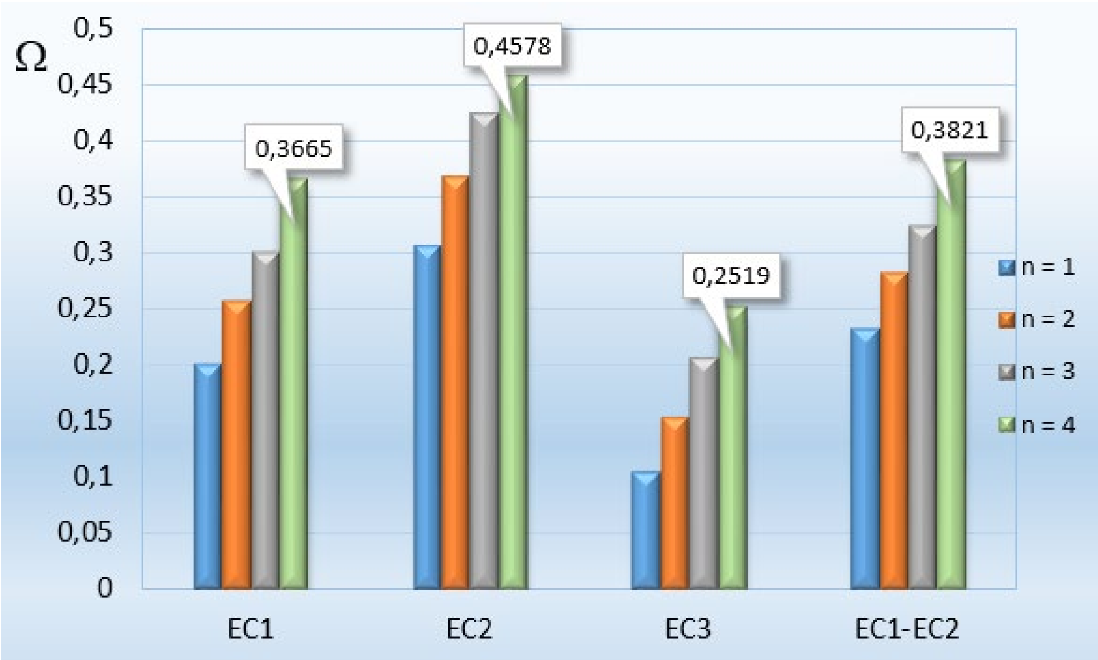

Figure 1 - Frequency parameter Ω for boundary conditions with *n* = 1 – 4

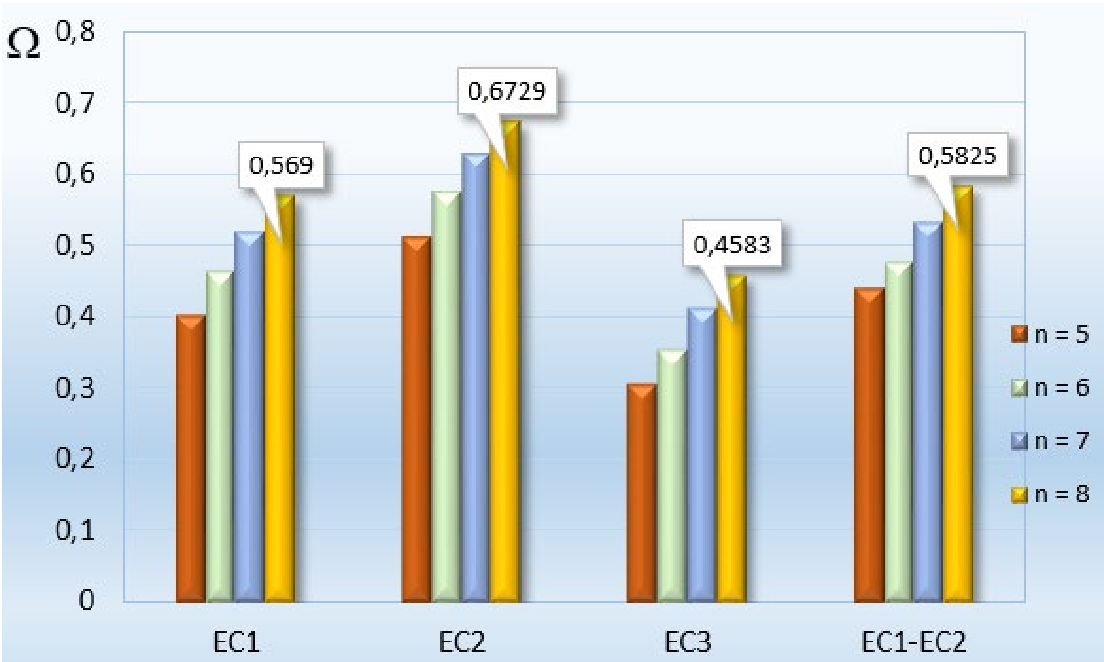

Figure 2 - Frequency parameter Ω for boundary conditions with *n* = 5 – 8

ISBN 978-3-98924-070-4

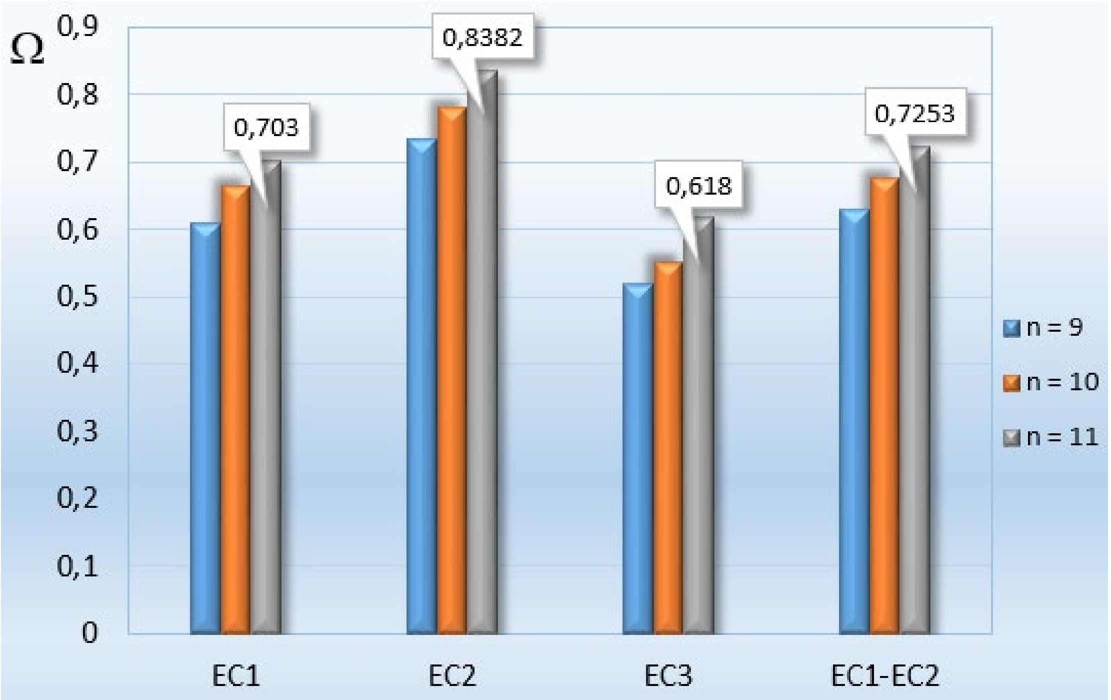

Figure 3 - Frequency parameter Ω for boundary conditions with *n* = 9 – 11

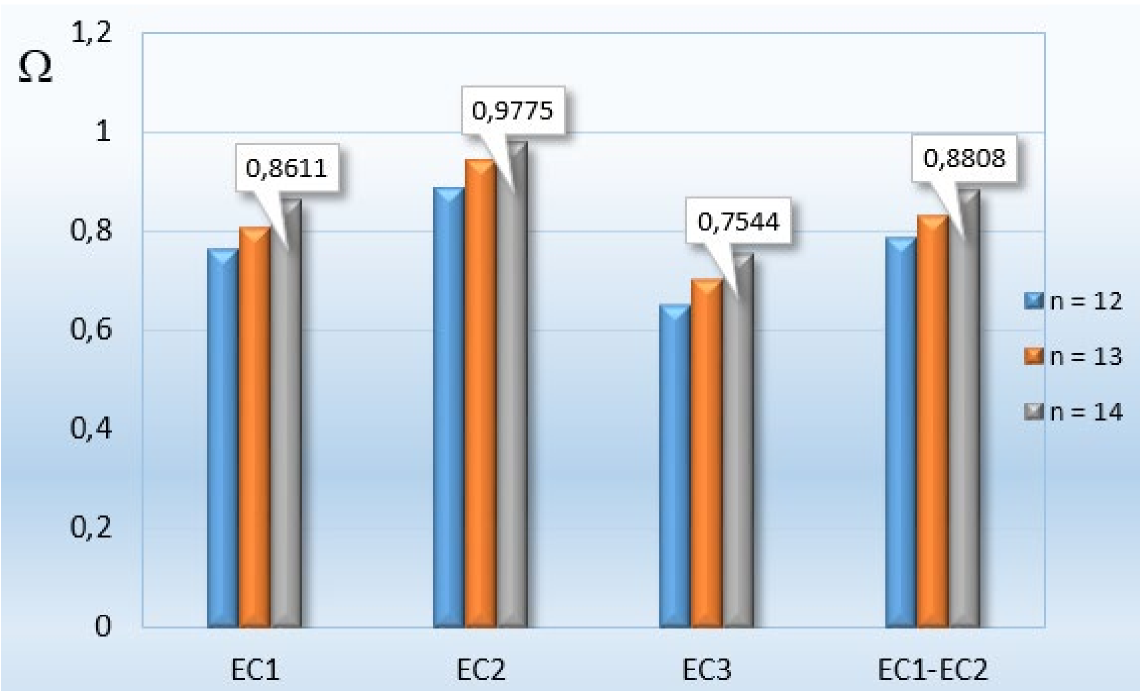

Figure 4 - Frequency parameter Ω for boundary conditions with *n* = 12 – 14

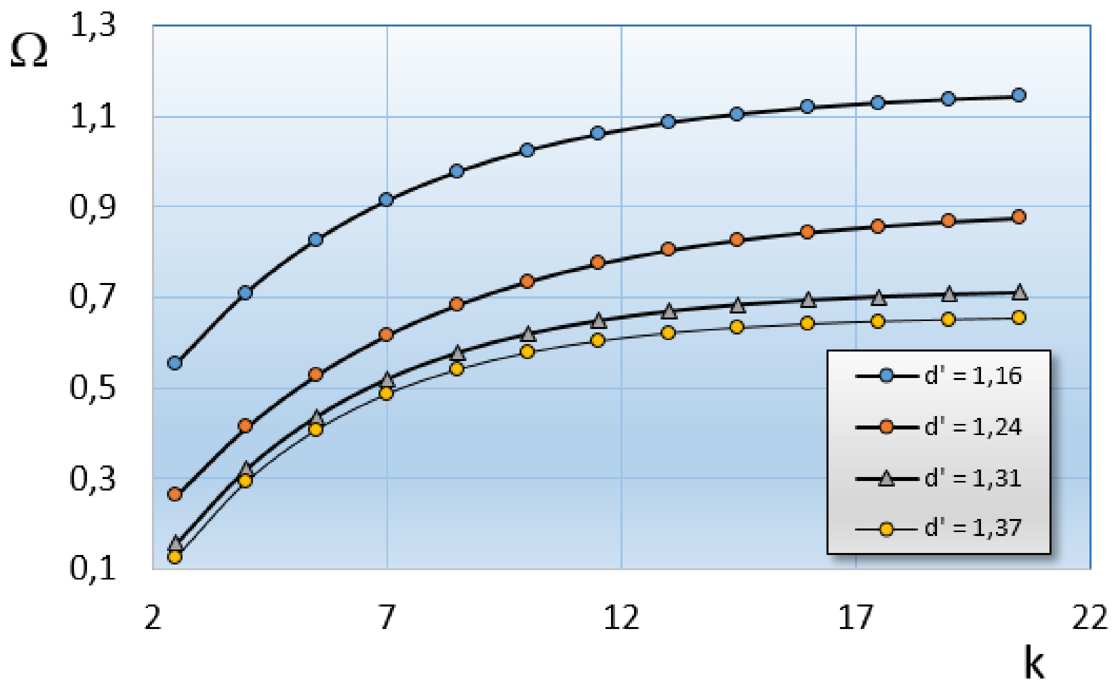

Figure 5 - Dependence Ω = Ω (k) for fixed length-to-radius ratios *d′*

The values of characteristic frequency Ω for the vibrations of cylindrical shells of laminated composites depending on the wave number n are shown in Figures 1 - 4. The dependence $\Omega = \Omega\ (k)$ for fixed length-to-radius ratios $d' = L/R$ (*EC* 2 boundary condition) is shown in Figure 5.

Summary and conclusions.

A computational model based on the Haar wavelet discretization method was applied to the analysis of free vibrations of composite laminated cylindrical shells. The vibrations of a laminated composite sample occurred under different boundary conditions. The characteristics of mechanical vibrations were calculated based on the classical shell theory. The discretization method of the control equations and the corresponding boundary conditions was implemented using discrete wavelet transforms. It was found that boundary conditions, length-to-radius ratios, lamination schemes, and elastic moduli ratios affect the natural frequency parameters of cylindrical shells made of laminated composites. The discrete wavelet analysis technique can also be used to describe vibrations of thick composite laminated and functionally graded shells.