KAPITEL 7 / *CHAPTER 7* [7]
LAMB WAVES PROPAGATION IN LAYERED STRUCTURES

DOI: 10.30890/2709-2313.2025-38-02-008

Introduction

Non-destructive testing, as well as structural health monitoring, determine the integrity and degradation of composite structures to ensure their operability. The working object of active diagnostics is the ultrasonic transient wave. In order to detect damage, localize and subsequently evaluate damage, understanding the wave propagation characteristics of composites is essential for the successful application of diagnostic methods [1, 2].

The effects of wave propagation in composites are complex due to the nature of the component heterogeneity, inherent material anisotropy and multilayer construction. These features are the reason why the wave mode velocity is macroscopically dependent on the laminate layup, the direction of wave propagation, frequency and interface conditions.

The propagation of elastic waves in isotropic plate structures is characterized by repeated reflections on the upper and lower surfaces alternately. As a result, the propagation of waves from their mutual interference is guided by the surfaces of the plates. The guided wave can be modeled by imposing surface boundary conditions on the equations of motion.

The effects of wave propagation in composite structures are accompanied by the phenomenon of dispersion, i.e. the propagation velocity of a guided wave along the plate is a function of frequency or, equivalently, wavelength [3, 4]. In particular, guided waves propagating along the plane of an elastic plate with tension-free boundaries are called Lamb waves. Since guided waves remain confined within the structure, they can propagate over large distances, allowing a large area to be surveyed with only a limited number of sensors.

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This property makes them well suited for continuous monitoring techniques for ultrasonic testing of entire structures and their elements, which are used in various industrial fields. In isotropic plates, guided waves can be classified into three types according to their polarization (direction of the displacement vector). Lamb waves polarized in the plane perpendicular to the plate, in the *x*-*z* plane, are called symmetric (or longitudinal, S) waves and antisymmetric (or flexural, A) waves, while those polarized in the plane of the plate (along the y-axis) are called shear horizontal (SH) waves [5, 6]. SH waves can also be either symmetric or antisymmetric about the mid-plane. S and A waves are controlled by plane strain state (displacements *u* and *w*); while SH waves are controlled by antiplane strain (displacement *v* only).

Conventionally, S_n and A_n with index $n = 0, 1, 2, 3...$ represent symmetric and antisymmetric Lamb wave modes, respectively; SHn with even and odd index n denote symmetric and antisymmetric SH waves, respectively.

Wave interactions of waves propagating in multilayer composites depend on the properties of the components, geometry, direction of propagation, frequency, and conditions at the interface. For the case when the wavelengths are significantly larger than the dimensions of the composite components (fiber diameters and spacing), each plate can be considered as a sample made of an equivalent homogeneous orthotropic or transversely isotropic material. Such a material is characterized by an axis of symmetry parallel to the fibers.

Scattering of tensile waves was recorded in experimental studies under conditions when the wavelength had the same order of magnitude as the fiber diameter. It should be noted that for flexural waves, scattering appeared when the wavelength was more than an order of magnitude greater than the fiber diameter.

Composite laminates consist of macroscopically homogeneous layers. In this case, wave interactions include not only reflection at the surfaces, but also reflection and refraction between the layers, manifested in the form of resultant waves. These interfering wave packets propagate along the plane of the plate.

The process of Lamb wave propagation in the composite is characterized by the following features. The velocity of the wave packet depends on the direction of its propagation. In addition, a consequence of elastic anisotropy is the loss of pure wave modes. The dependence of the wave velocity on the direction of propagation implies that the direction of the group velocity in general does not coincide with the wave vector (or wave normal).

The distinction between the wave mode types in composites is rather arbitrary. The reason for this is that the three wave mode types are usually related. Engineering practice usually uses symmetric laminates when designing composite structures. Lamb waves in symmetric laminates can be divided into symmetric and antisymmetric modes. For the symmetric modes, one type is designated as quasi-extended. In this case, the dominant component of this symmetric mode of the polarization vector is located along the direction of propagation. The second type of symmetric mode is quasi-horizontal shear. For quasi-horizontal shear, the polarization vector is mainly parallel to the plane of the plate.

In exactly the same way, quasi-flexural and quasi-horizontal shears are generated for antisymmetric types of wave modes. In theoretical analysis, two approaches can be distinguished for the study of Lamb waves in composites. The first method is associated with exact solutions according to the three-dimensional theory of elasticity. The second method is characterized by the inclusion of approximate solutions according to theories of plates.

The dispersion relations of Lamb waves in a composite plate can be obtained by analyzing the mechanical elasticity in three dimensions. Further extension of the transfer matrix methodology in different types of composites was used to obtain the dispersion curves. The exact solutions of Lamb wave dispersion in composite shells are compared with the results of the Flugge shell theory. It should be noted that the scope of applicability of this methodology is limited only to the case of dispersion relations of phase, but not group velocity. Calculation of dispersion characteristics of multilayer composites is quite complex due to the presence of transcendental equations. In addition, such calculations should take into account the transient wave response of composites. Lamb wave packets that propagate in composites along an arbitrary direction generally cause a perturbation in the mechanical strain field. Such a change

 ISBN 978-3-98924-083-4

in the field characteristics involves three components of displacement, i.e., a generalized plane strain arising from the anisotropy of the material.

The displacement of plane harmonic waves can be described in general using 3-D elasticity. The initial stage of the analysis investigates the characteristics of Lamb waves in a single plate (monoclinic plate). In this case, a compact closed dispersion relation can be obtained by separating symmetric and antisymmetric modes using trigonometric functions through the plate thickness. Special cases are when the waves propagate along the symmetry axis of the material in such a way that mutual separation of S- and A waves as well as SH waves is considered. The final stage of the analysis generates a modified exponential form in the thickness direction to derive the dispersion relation for the composite laminate with special emphasis on symmetric laminates.

7.1. Propagation perpendicular to the mid-plane of the composite laminate

The propagation of wave packets is considered in a Cartesian coordinate system with the z-axis perpendicular to the mid-plane of the composite laminate spanned by the x- and y-axes. The two outer surfaces of the laminate are defined by the coordinates $z = \pm h/2$.

An arbitrary direction θ of the Lamb wave packet is defined counterclockwise with respect to the x-axis. In this case, a fixed layer of the composite laminate with an arbitrary orientation in the global coordinate system (x, y, z) is considered as a monoclinic material having x–y as a plane of symmetry. This fact causes the stress–strain relationships to take the following matrix form:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{21} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{31} & C_{32} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{54} & C_{55} & 0 \\ C_{61} & C_{62} & C_{63} & 0 & 0 & C_{66} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}. \quad (1)$$

ISBN 978-3-98924-083-4

The case where the global coordinate system (x, y, z) does not coincide with the main coordinate system of the material (x', y', z) of each layer, but forms an angle φ with the x-axis is considered separately (Fig. 1). For such conditions, the stiffness matrix C_{ij} (i, j = 1, 2, 3, 6) in the system (x, y, z) can be obtained from the plate stiffness matrix C_{0ij} in the system (x_0, y_0, z) using the transformation matrix method. The composite material specimen is orthotropic or transversely isotropic with respect to the main axes of the material in (x_0 , y_0, z). The plate stiffness matrix C_{0ij} can be calculated from the plate material properties E_k, ν_{kl} and G_{kl} (k, l = 1, 2, 3).

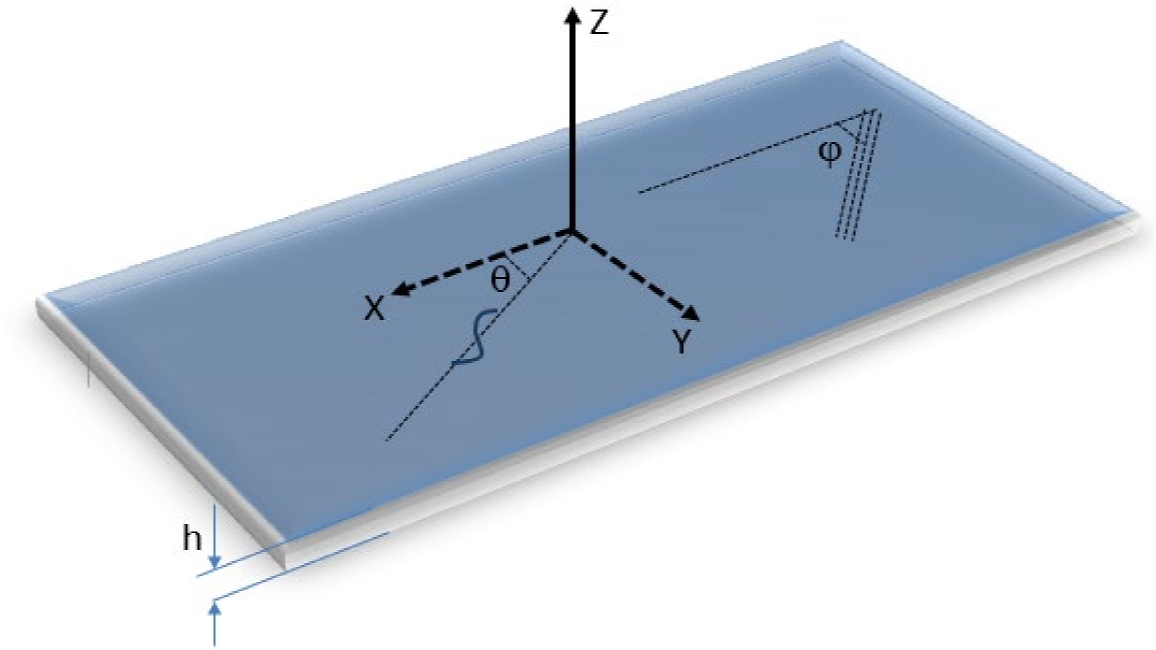

Figure 1 - Lamb wave propagation in a composite laminate

The relationships between deformations and displacements are as follows

$$\varepsilon_x = u_x,\quad \varepsilon_y = \upsilon_y,\quad \varepsilon_z = w_z,\quad \gamma_{yz} = \upsilon_z + w_y,$$

$$\gamma_{xz} = u_z + w_x,\quad \gamma_{xy} = u_y + \upsilon_x, \tag{2}$$

where

u is the displacement in the x direction;

υ is the displacement in the y direction;

w is the displacement in the z direction.

For the case of absence of external forces, the equations of motion can be

 ISBN 978-3-98924-083-4

expressed using the following relationships

$$\sigma_{xx} + \tau_{xy,y} + \tau_{xz,z} = \rho \ddot{u}, \tag{3}$$

$$\tau_{xy,x} + \sigma_{yy} + \tau_{yz,z} = \rho \ddot{\upsilon}, \tag{4}$$

$$\tau_{xz,x} + \tau_{yz,y} + \sigma_{zz} = \rho \ddot{w}, \tag{5}$$

where

ρ is the density of fixed lamina.

The boundary conditions on the upper and lower surfaces can be written using the following equation

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0, \quad z = \pm \frac{h}{2}. \tag{6}$$

Lamb wave packets propagate along the plane of a plate with boundaries free of additional mechanical stresses. On the other hand, Lamb waves are standing waves in the *z*-direction of the plate. Therefore, the wave motion can be expressed by a superposition of plane harmonic waves. Each plane harmonic wave propagating in the direction of the wave normal *k* can be described by the relation

$$\{u, \upsilon, w\} = \{U(z), V(z), W(z)\} \exp\{i[(k_x x + k_y y) - \omega t]\}, \tag{7}$$

where $k = [k_x, k_y]^T$.

Magnitude of *k* is

$$|k| = \sqrt{k_x^2 + k_y^2} = \frac{\omega}{c_p}, \tag{8}$$

$$k = \frac{2\pi}{\lambda}, \tag{9}$$

where

λ is the wavelength;

ω is the angular frequency;

c_p is the phase velocity.

Mechanical stresses in each layer are

$$\sigma_x = [C_{11} k_x U + C_{12} k_y - iC_{13} W' + C_{16}(k_y U + k_x V)] \times$$

 ISBN 978-3-98924-083-4

$$\times \exp\{i[(k_x x + k_y y) - \omega t]\}, \tag{10}$$

$$\sigma_y = [C_{12} k_x U + C_{22} k_y - iC_{23} W' + C_{36}(k_y U + k_x V)] \times$$
$$\times \exp\{i[(k_x x + k_y y) - \omega t]\}, \tag{11}$$

$$\sigma_z = [C_{31} k_x U + C_{32} k_y - iC_{33} W' + C_{63}(k_y U + k_x V)] \times$$
$$\times \exp\{i[(k_x x + k_y y) - \omega t]\}, \tag{12}$$

$$\tau_{yz} = [C_{44}(V' + ik_y W) + C_{45}(U' + ik_x W)] \times$$
$$\times \exp\{i[(k_x x + k_y y) - \omega t]\}, \tag{13}$$

$$\tau_{xz} = [C_{54}(V' + ik_y W) + C_{55}(U' + ik_x W)] \times$$
$$\times \exp\{i[(k_x x + k_y y) - \omega t]\}, \tag{14}$$

$$\tau_{xy} = [C_{61} k_x U + C_{62} k_y V - iC_{63} W' + C_{66}(k_y U + k_x V)] \times$$
$$\times \exp\{i[(k_x x + k_y y) - \omega t]\}. \tag{15}$$

The equations for mechanical displacements for an off-axis composite plate allow separation into symmetric (index "s") and antisymmetric (index "a") wave modes. This separation leads to a particularly simple form of the asymptotic representation

$$U_s = A_s \cos \xi z, \quad V_s = B_s \cos \xi z, \quad W_s = C_s \sin \xi z, \tag{16}$$

$$U_a = A_a \cos \xi z, \quad V_a = B_a \cos \xi z, \quad W_a = C_a \sin \xi z, \tag{17}$$

where ξ is the fixed variable.

7.2. Kinetics of symmetric and antisymmetric modes of Lamb waves

The method of analyzing the resulting system of equations for mechanical displacements and stresses can be divided into two successive stages. In the first approximation, only symmetrical modes of Lamb waves are subjected to theoretical analysis during their group motion along the anisotropic medium, which constitutes the volume of the laminated composite. In addition, a set of compact dispersion relations is analyzed separately for both symmetric and antisymmetric Lamb wave modes. This analysis is performed using metric functions for the corresponding wavelet transform for all points in the laminated composite volume. Symmetrical modes are considered first. At the second stage, the entire sequence of solutions of this system is transformed into a matrix form

$$\begin{bmatrix} \Gamma_{11}-\rho\omega^2 & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22}-\rho\omega^2 & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33}-\rho\omega^2 \end{bmatrix} \begin{Bmatrix} A_s \\ B_s \\ C_s \end{Bmatrix} = 0 . \quad (18)$$

Matrix elements can be expressed by the following expressions

$$\Gamma_{11} = C_{11}k_x^2 + 2C_{61}k_xk_y + C_{66}k_y^2 + C_{55}\xi^2 , \quad (19)$$

$$\Gamma_{12} = C_{61}k_x^2 + (C_{12}+C_{66})k_xk_y + C_{62}k_y^2 + C_{45}\xi^2 , \quad (20)$$

$$\Gamma_{13} = -i\left[(C_{31}+C_{55})k_x + (C_{63}+C_{45})k_y\right]\xi , \quad (21)$$

$$\Gamma_{22} = C_{66}k_x^2 + 2C_{26}k_xk_y + C_{22}k_y^2 + C_{44}\xi^2 , \quad (22)$$

$$\Gamma_{23} = -i\left[(C_{36}+C_{45})k_x + (C_{23}+C_{44})k_y\right]\xi , \quad (23)$$

$$\Gamma_{33} = C_{55}k_x^2 + 2C_{45}k_xk_y + C_{44}k_y^2 + C_{33}\xi^2 . \quad (24)$$

A similar technique is consistently applied to the antisymmetric mode. As a result, the resulting matrix becomes equal to ($\mathbf{\Gamma} - \rho\omega^2\mathbf{I}$), where **I** is the identity matrix. For the case when the Hermitian matrix **Γ** is positive definite, it can be shown that the eigenvalues of the symmetric and antisymmetric modes coincide.

The non-trivial solutions A_s, B_s and C_s participate in the zeroing of the determinant of the characteristic matrix ($\mathbf{\Gamma} - \rho\omega^2\mathbf{I}$), which yields the following sixth-order polynomial in ξ

$$\xi^6 + \alpha_1\xi^4 + \alpha_2\xi^2 + \alpha_3 = 0 , \quad (25)$$

where α_i are real-valued coefficients of C_{ij}, k, and $\rho\omega^2$.

Analysis of the system of characteristic equations showed that there are three solutions. The properties of these solutions include their positivity, nonzero difference, and discreteness. As a rule, such solutions are related to indices n_j (j = 1, 2, 3). For each index nj in the symmetric modes of the wave packets B_s and C_s, which are related to the symmetric modes of Lamb waves in composite structures, the following relationships can be written in terms of As

$$B_s = \frac{\left(\Gamma_{11} - \rho\omega^2\right)\Gamma_{23} - \Gamma_{12}\Gamma_{13}}{\Gamma_{13}\left(\Gamma_{22} - \rho\omega^2\right) - \Gamma_{12}\Gamma_{23}} A_s = RA_s, \tag{26}$$

$$C_s = \frac{\Gamma_{12}^2 - \left(\Gamma_{11} - \rho\omega^2\right)\left(\Gamma_{22} - \rho\omega^2\right)}{\Gamma_{13}\left(\Gamma_{22} - \rho\omega^2\right) - \Gamma_{12}\Gamma_{23}} A_s = iSA_s. \tag{27}$$

As a result, the modified equations for mechanical shears and stresses will have the form

$$\left(\sigma_z, \tau_{yz}, \tau_{xz}\right)\Big|_{z=h/2} = \sum_{j=1}^{3} [H_{1j} \sin\left(\xi_j z + \varphi\right), H_{2j} \cos\left(\xi_j z + \varphi\right) + \\ + H_{3j} \cos\left(\xi_j z + \varphi\right)] A_j = 0, \tag{28}$$

where φ = 0 and π/2 represent anti-symmetric and symmetric Lamb wave modes, and

$$H_{11}(H_{22}H_{33} - H_{23}H_{32})\tan\left(\frac{\xi_1 h}{2} + \varphi\right) + \\ + H_{12}(H_{23}H_{31} - H_{21}H_{33})\tan\left(\frac{\xi_2 h}{2} + \varphi\right) + \\ + H_{13}(H_{21}H_{32} - H_{22}H_{31})\tan\left(\frac{\xi_3 h}{2} + \varphi\right) = 0. \tag{29}$$

The numerical calculation methodology for Lamb wave propagation in composite structures assumes that the interfaces between layers are ideally coupled. For each layer, the displacement components in the corresponding z-axis equation must be modified into exponential forms to account for the inhomogeneity of the multilayer laminate.

$$U = A\exp\left(i\xi z\right), \qquad V = B\exp\left(i\xi z\right), \qquad W = -iC\exp\left(i\xi z\right). \tag{30}$$

 ISBN 978-3-98924-083-4

The general solution in each lamina is

$$\{U,V,W\} = \exp\{i\,[(k_x x + k_y y) - \omega t]\}\cdot \sum_j A_j \{1, R_j, S_j\} \exp(i\xi_j z) \quad . \qquad (31)$$

Symmetrical and asymmetrical wave modes in conventional laminates cannot be separated. It should be noted that symmetrical laminates are used in engineering practice when designing composite structures. A reliable method for separating the two types of wave modes is to generate boundary conditions on both the upper and middle planes of the surface. For the upper boundary of the laminate, the boundary conditions can be written as follows

$$\{\sigma_z, \tau_{yz}, \tau_{xz}\}\Big|_{z=h/2} = 0 \quad . \qquad (32)$$

The symmetry conditions for the entire laminate allow only half of the entire sample to be analyzed. In a subsequent step, the following conditions are imposed on the stress and displacement components in the mid-plane for symmetric modes

$$\{u, \upsilon, \sigma_z\}\Big|_{z=0} = 0 \quad . \qquad (33)$$

The implicit functional form G (ω, $\boldsymbol{k}$) = 0, or G (ω, k, θ) = 0 can be used to formulate the dispersion relation between ω and k. This dispersion relation can be explicitly solved in the form of real roots ω = W ($\boldsymbol{k}$), or ω = W (k, θ).

The number of possible solutions with different functions W tends to infinity. Such solutions correspond to different wave modes. The phase velocity vector for plane modes is defined as $c_p = (\omega/k)\cdot(\mathbf{k}/|\mathbf{k}|) = (\omega/k^2)\mathbf{k}$. Therefore, its magnitude is $c_p = \omega/k$. The set of all statistical samples k from the origin for c_p at a given frequency forms the so-called velocity curve. The radius vectors of the velocity curves in the direction of a given k represent the admissible dispersion of the phase velocity of the different wave modes.

ISBN 978-3-98924-083-4

Summary and conclusions.

In this study, a methodology for describing the propagation of symmetric and antisymmetric Lamb waves in laminated structures was proposed. Preliminary analysis showed that it is more convenient to consider the propagation of wave packets in a Cartesian coordinate system in the direction perpendicular to the middle plane of the composite laminate. For the case of a fixed direction of propagation of a wave packet in a laminated composite, the latter can be represented as a monoclinic material with a plane of symmetry. This fact made it possible to represent the stress-strain relationships in matrix form. The Lamb wave packet propagation model included several boundary conditions. In the direction perpendicular to the main cross-section, the wave field characteristics are well described by the standing wave model. Therefore, the wave motion can be expressed as a superposition of plane harmonic waves. It is shown that the structure of numerical calculations can be divided into two stages. At the first stage, only symmetric modes of Lamb waves are analyzed during their group motion along the anisotropic medium constituting the volume of the layered composite. For asymmetric modes, a set of compact dispersion relations was analyzed. This analysis was performed using metric functions for the corresponding wavelet transform for all points of the composite volume.

 ISBN 978-3-98924-083-4