KAPITEL 10 / CHAPTER 10 [10]
LAYER-BY-LAYER WAVE-PACKET PROPAGATION IN STRESS AND DISPLACEMENT FIELDS OF LAMINATED STRUCTURES

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Introduction

Laminated composite structures are increasingly employed in advanced engineering systems due to their high strength-to-weight ratios, tailored mechanical properties, and design adaptability. These materials, consisting of multiple bonded layers with distinct elastic characteristics, are fundamental in aerospace, civil infrastructure, architecture, and energy systems [1]. However, their layered nature also leads to a complex internal stress environment, particularly when subjected to dynamic loading. Understanding how stress and displacement fields evolve under wave propagation is critical not only for predicting performance but also for detecting internal flaws, delaminations, or structural degradation.

The propagation of mechanical waves in laminated media is a multi-scale phenomenon characterized by interactions at interlayer boundaries, internal reflections, mode conversions, and dispersion [2, 3]. Of particular interest is the behavior of wave packets—localized transient disturbances that can be induced by impact, acoustic excitation, or other short-duration mechanical loads. These packets traverse the layered structure and interact with each layer's material properties and geometric interfaces, generating highly non-uniform and anisotropic stress and displacement fields. Capturing this complexity requires analytical tools that can accurately resolve wave dynamics at both the global and local levels, while maintaining mathematical and computational tractability.

Traditional wave propagation models, especially those based on modal decomposition or single-layer assumptions, often fall short in dealing with strongly coupled multilayered systems [4, 5]. For this reason, advanced modeling techniques have been developed to account for interfacial behavior and full-field responses.

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Among them, two complementary methods have gained significant prominence in the study of multilayer wave propagation: the global matrix concept and the two-dimensional Fourier transform technique.

The global matrix concept (GMC) is a powerful tool for analyzing layered structures by formulating the overall problem as a system of equations that incorporates all boundary and interfacial conditions simultaneously [6]. In this approach, each layer’s mechanical behavior is described by a set of state variables (typically displacements and stresses), and the continuity conditions at interfaces are rigorously enforced. The result is a global matrix system that captures the full dynamics of the entire multilayer stack. The GMC is particularly advantageous for studying steady-state and time-harmonic responses in stratified media, and it has been extensively applied in fields such as elasto-dynamics, thermos-elasticity, and piezo-elastic wave analysis.

What sets the GMC apart is its ability to handle arbitrary stacking sequences, anisotropic materials, and complex boundary conditions in a unified framework. Unlike recursive techniques, which build layer-by-layer transfer functions, the GMC assembles the global behavior in a monolithic sample, which enhances numerical stability and analytical clarity, especially for thick or highly heterogeneous laminates. The method’s effectiveness has been demonstrated in problems ranging from guided wave propagation in plates and shells to the scattering of waves in layered half-spaces. Nonetheless, the GMC often requires coupling with transform methods to handle transient, localized phenomena, which leads to the second pillar of this study.

The two-dimensional Fourier transform technique (2D-FTT) offers a natural and elegant method for analyzing time-space dependent wave phenomena in continuous media [7]. By transforming the governing partial differential equations in both spatial and temporal dimensions, one obtains algebraic equations in the frequency-wavenumber domain. This transformation simplifies the mathematical structure of the problem and enables the characterization of wave dispersion, group velocity, and frequency-dependent behavior with high resolution. For laminated media, where the propagation characteristics are often highly dispersive and anisotropic, the 2D-FTT

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provides a precise framework for dissecting the frequency content and directional behavior of wave packets.

In the context of laminated structures, combining the GMC with the 2D-FTT enables a comprehensive analysis that bridges global structural behavior and localized transient effects. The GMC provides the layered system’s response to harmonic inputs in the transformed domain, while the 2D-FTT allows the reconstruction of time- and space-resolved fields through inverse transformations. This hybrid methodology is particularly suited for analyzing short-duration excitations, such as impact-induced wave packets or localized ultrasonic pulses, where both frequency and spatial resolution are critical. Moreover, it facilitates the study of complex interactions such as interfacial stress concentrations, wave trapping in specific layers, and sensitivity to layer imperfections.

The significance of this combined approach is not limited to theoretical interest. It has practical implications in non-destructive evaluation (NDE), structural health monitoring (SHM), and material characterization. For example, guided wave inspection techniques rely on accurate modeling of wave propagation through complex multilayered paths, and errors in estimating stress or displacement fields can lead to misinterpretation of diagnostic signals. Similarly, in high-performance composite components, predicting the response to dynamic loads—such as impacts or vibrations—requires a model that captures the full wave dynamics across all layers.

Despite the wide application of both the global matrix concept and Fourier techniques individually, their integrated use in the time-space characterization of wave packets in laminated media has not been fully exploited. Existing studies often rely on simplifications such as scalar wave approximations, quasi-static assumptions, or ignore the interaction of multiple wave modes. The present work seeks to overcome these limitations by constructing a rigorous mathematical model that combines GMC and 2D-FTT to yield a full-field, layer-by-layer description of stress and displacement wave propagation.

Specifically, this study develops a theoretical and computational framework that models the transient response of laminated structures to wave-packet excitation using

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the global matrix concept formulated in the frequency-wavenumber domain. The displacement and stress components in each layer are determined by solving the global matrix system for each spectral component, followed by an inverse two-dimensional Fourier transform to reconstruct the physical fields in time and space. This approach allows for accurate resolution.

10.1. Global matrix concept

The global matrix model assumes a detailed description of the displacement field and mechanical stresses in the entire layered structure using the example of a laminated composite. The control equations essentially represent a system of nonlinear relationships between the elements of the displacement field $U = U(u_i)$, stresses $W = W(\sigma_j)$ and matrix components $K_{ij} = K_{ij}(c_{ij})$. The roots of the control equations correspond to pairs of quasi-longitudinal and quasi-transverse components of wave packets. The tensor coefficients χ_q determine the diagonal elements of the global displacement matrix of the laminar composite.

The displacements (u_1, u_2, u_3) can be represented as functions of unit amplitude (U_{1q}), by defining the ratios of the displacement components as $V_q = U_{2q}/U_{1q}$ and $W_q = U_{3q}/U_{1q}$, namely

$$V_q = \frac{K_{11}(\chi_q)K_{23}(\chi_q) - K_{12}(\chi_q)K_{13}(\chi_q)}{K_{13}(\chi_q)K_{22}(\chi_q) - K_{12}(\chi_q)K_{23}(\chi_q)}, \qquad (1)$$

$$W_q = \frac{K_{11}(\chi_q)K_{23}(\chi_q) - K_{12}(\chi_q)K_{13}(\chi_q)}{K_{33}(\chi_q)K_{12}(\chi_q) - K_{13}(\chi_q)K_{23}(\chi_q)}. \qquad (2)$$

The total displacement in terms of V_q and W_q are

$$u_1 = \sum_{q=1}^{6} U_{1q} \exp\left[i\left(k_1 x_1 + k_2 x_2 + \chi_q x_3 - \omega t\right)\right], \qquad (3)$$

$$u_2 = \sum_{q=1}^{6} V_q U_{1q} \exp\left[i\left(k_1 x_1 + k_2 x_2 + \chi_q x_3 - \omega t\right)\right], \qquad (4)$$

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$$u_3 = \sum_{q=1}^{6} W_q U_{1q} \exp\left[i\left(k_1 x_1 + k_2 x_2 + \chi_q x_3 - \omega t\right)\right] \quad . \quad (5)$$

And the total stress can be simplified as

$$\sigma_{33} = i\sum_{q=1}^{6} D_{1q} U_{1q} \exp\left[i\left(k_1 x_1 + k_2 x_2 + \chi_q x_3 - \omega t\right)\right] \quad , \quad (6)$$

$$\sigma_{13} = i\sum_{q=1}^{6} D_{2q} U_{1q} \exp\left[i\left(k_1 x_1 + k_2 x_2 + \chi_q x_3 - \omega t\right)\right] \quad , \quad (7)$$

$$\sigma_{23} = i\sum_{q=1}^{6} D_{3q} U_{1q} \exp\left[i\left(k_1 x_1 + k_2 x_2 + \chi_q x_3 - \omega t\right)\right] \quad , \quad (8)$$

where

$$D_{1q} = k_1\left(c_{13} + c_{36}V_q\right) + k_2\left(c_{36} + c_{23}V_q\right) + c_{33}W_q\chi_q \, , \quad (9)$$

$$D_{2q} = c_{55}W_q k_1 + c_{45}W_q k_2 + \chi_q\left(c_{55} + c_{45}V_q\right) , \quad (10)$$

$$D_{3q} = c_{45}W_q k_1 + c_{44}W_q k_2 + \chi_q\left(c_{45} + c_{44}V_q\right) . \quad (11)$$

The layers of composite structures consist of a linear elastic material with perfectly bonded interfaces with a continuous strain distribution and that the stresses at each interface are equal. The composite fiber aggregate is sufficiently rigidly bonded to the matrix. There is stress/strain compatibility at the fiber-matrix interface.

The most commonly used methods for generating Lamb wave dispersion curves of layered anisotropic media are based on three-dimensional linear elasticity in combination with global matrix and transfer matrix approaches. This technique uses only the global matrix approach, since the transfer matrix is considered stable only for the low-frequency thickness product.

The global matrix concept analysis methodology is based on examining all the equations from each layer to form a single global matrix (Fig. 1). This matrix describes the displacement and stress fields associated with wave propagation. The global matrix method consists of $X\,(n - 1)$ equations for n layers, where X represents the number of expected partial waves. This method is robust and remains stable for any frequency-thickness product, since it does not rely on wave coupling from one interface to another.

For the *k*-th layer of a monoclinic plate with thickness d*k*, the displacement u_i and stress r_{ij} can be written as follows

$$(u_1, u_2, u_3)_k = \left[\sum_{k=1}^{6} (1, V_q, W_q) U_{1q} \exp\{i(k_1 x_1 \sin\varphi + k_2 x_2 + \chi_q x_3 - \omega t)\} \right]_k, \quad (12)$$

$$(\sigma_{33}, \sigma_{13}, \sigma_{23})_k = \left[i \sum_{k=1}^{6} (D_{1q}, D_{2q}, D_{3q}) U_{1q} \exp\{i(k_1 x_1 \sin\varphi + k_2 x_2 + \chi_q x_3 - \omega t)\} \right]_k, \quad (13)$$

where φ is the incident angle relative to the *x*-axis.

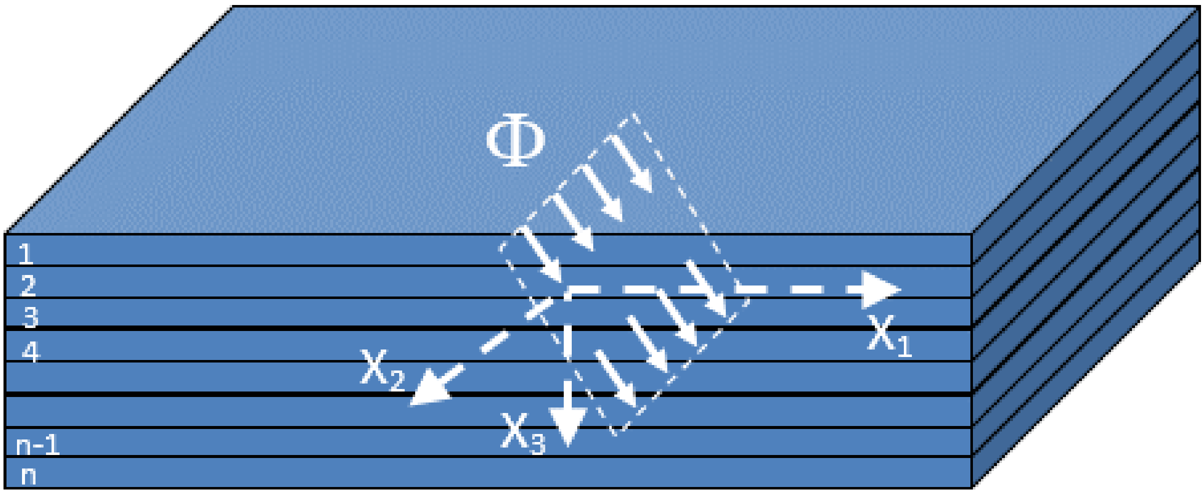

Figure 1 - N-layered composite laminate

By linking together, the matrix elements for mechanical stress, shear, and the energy component of a wave packet as it moves in a fixed direction within a laminated composite, the following matrix equation can be written

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \sigma_{33} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix}_k = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ V_1 & V_1 & V_3 & V_3 & V_5 & V_5 \\ W_1 & -W_1 & W_3 & -W_3 & W_5 & -W_5 \\ iD_{11} & iD_{11} & iD_{13} & iD_{13} & iD_{15} & iD_{15} \\ iD_{21} & -iD_{21} & iD_{23} & -iD_{23} & iD_{25} & -iD_{25} \\ iD_{31} & -iD_{31} & iD_{33} & -iD_{33} & iD_{35} & -iD_{35} \end{bmatrix}_k \times$$

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$$\times \begin{Bmatrix} U_{11}\exp[i(k_1x_1\sin\varphi + k_2x_2 + \chi_1x_3 - \omega t)] \\ U_{12}\exp[i(k_1x_1\sin\varphi + k_2x_2 + \chi_1x_3 - \omega t)] \\ U_{13}\exp[i(k_1x_1\sin\varphi + k_2x_2 + \chi_1x_3 - \omega t)] \\ U_{14}\exp[i(k_1x_1\sin\varphi + k_2x_2 + \chi_1x_3 - \omega t)] \\ U_{15}\exp[i(k_1x_1\sin\varphi + k_2x_2 + \chi_1x_3 - \omega t)] \\ U_{16}\exp[i(k_1x_1\sin\varphi + k_2x_2 + \chi_1x_3 - \omega t)] \end{Bmatrix}. \quad (14)$$

The part of the matrix equation (14) that contains the displacement and stress vectors is denoted as P_k. Accordingly, the right-hand side is denoted as X_k, the displacement amplitude is U_k, and the wave equation is denoted as D_k. In this case, the short form of the matrix equation is

$$\{P_k\} = [X_k]\cdot[D_k]\cdot\{U_k\}. \quad (15)$$

Each layer of the monoclinic plate of the laminar composite has six partial waves, denoted as (L+/), (SV+/) and (SH+/). These Lamb waves can be interpreted as quasi-longitudinal, quasi-shear vertical and quasi-shear horizontal waves, respectively. Positive and negative signs represent downward and upward traveling waves. The characteristic equations of Lamb waves can be written as a result of the analysis of displacements and shears in the second interface (*i*2), which consists of the lower surface of layer 2 (*l*2) and the upper surface of layer 3 (*l*3).

Mechanical displacements and shifts in a fixed layer are subject to a system of equations

$$\{P_{l2,b}\} = [[X_{l2,b}][D_{l2,b}]\{U_{l2,b}\}], \quad (16)$$

$$\{P_{l3,b}\} = [[X_{l3,b}][D_{l3,b}]\{U_{l3,b}\}]. \quad (17)$$

The condition of continuity of displacement within one layer has the form

$$[[Z_{l2,b}][-Z_{l3,b}]]\begin{Bmatrix}\{U_{l2,b}\} \\ U_{l3,t}\end{Bmatrix} = \{0\}, \quad (18)$$

where indexes “b” and “t” are the nearest surfaces of fixed layer.

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The global matrix combines the conditions for all layers and the five partial Lamb waves that propagate in the monoclinic laminate

$$\begin{bmatrix} \lfloor Z_{l1,b} \rfloor & \lfloor -Z_{l2,t} \rfloor & [0] & [0] & [0] \\ [0] & \left[Z_{l2,b}\right] & \left[-Z_{l3,t}\right] & [0] & [0] \\ [0] & [0] & \left[X_{l3,b}\right] & \left[-Z_{l4,t}\right] & [0] \\ [0] & [0] & [0] & \left[X_{l4,b}\right] & \left[-Z_{l5,t}\right] \end{bmatrix} = \{0\} \quad . \tag{19}$$

The boundary conditions for Lamb waves specify that the mechanical stresses on the top and bottom surfaces are zero. The solution to this boundary condition can be obtained by splitting the submatrix of the top and bottom layers in the equation into their associated mechanical stresses and displacements. This split allows a new matrix to be formed in the next step.

In the next step after the separation, only the components of mechanical stress are analyzed to obtain the dispersion curves of Lamb waves. Assuming that the displacement components from the upper and lower layers of the equation are negligible, the newly obtained matrix can be separated.

10.2. Two-dimensional Fourier transform technique

The Lamb wave sets were averaged during processing to reduce noise. The resulting waveform was analyzed using the Fast Fourier Transform. The continuous wavelet transform was implemented using the complex Morlet wavelet. The Fourier spectrum was characterized by a large number of peaks due to the frequency dispersion of the Lamb waves in the laminate.

To identify the Lamb wave modes, theoretical dispersion curves of Lamb wave packets were calculated. The group velocities of all modes propagating in a quasi-isotropic laminate of fixed thickness were calculated for each frequency. The time for symmetric and antisymmetric modes was determined from the phase and group velocities. The corresponding dispersion curves are shown in Fig. 2.

The interaction of waves with laminar composite bulk delamination and lateral surface effects is fairly easy to detect in the time-space wave field. However, the

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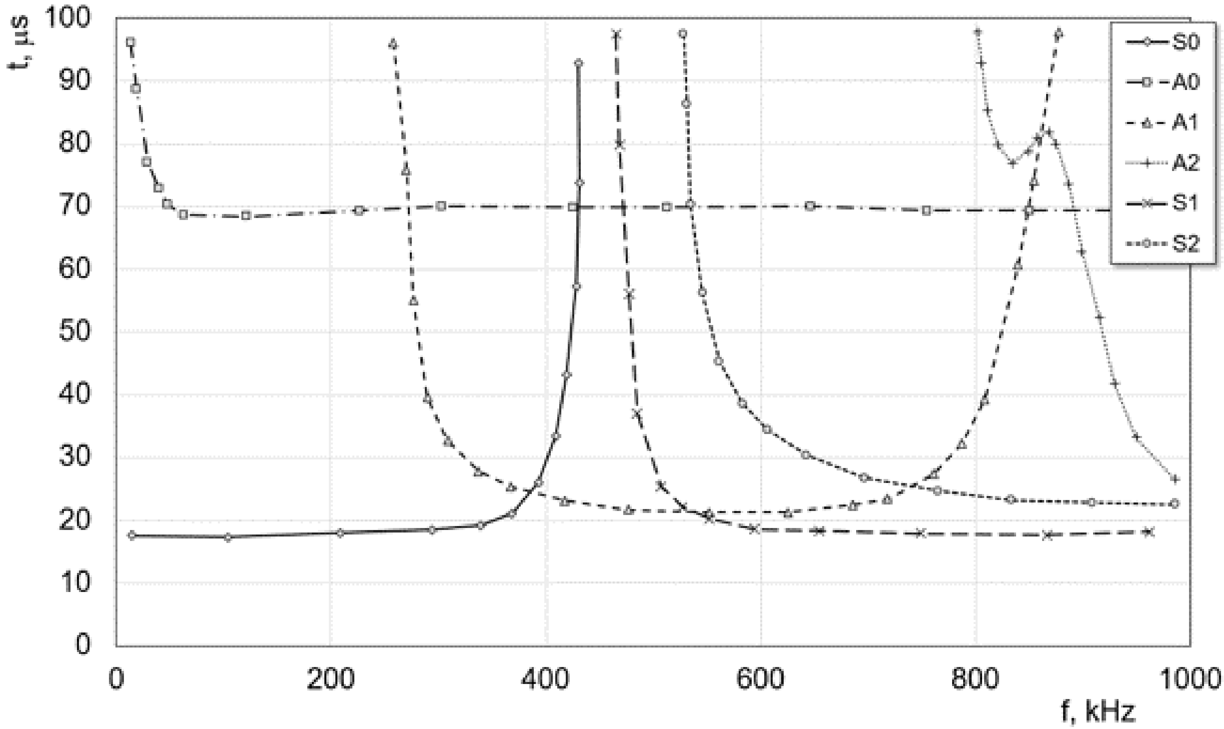

Figure 2 - Dispersion curves for all modes of Lamb waves

implicit characteristics of wave packet propagation, such as the content of the wave mode and how the modes change along the wave path, are not easily visible.

Wave data analysis, representing the characteristics of a wave number set by the locus of wave front points, has abundant information regarding the existence of different wave modes and wave propagation characteristics.

Localization of a two-dimensional Fourier transform, where the wave number is a function of distance, can be realized by methods of transforming the wave field into a time-space representation of the wave number.

Spectral analysis of Lamb waves in time and space using a two-dimensional Fourier transform allows one to write the frequency-wave number representation (f - k) as

$$V(f,k)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v(t,x)\exp[-i(2\pi f t-kx)]dt\,dx \quad , \qquad (20)$$

where

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$V(f, k)$ is the resulting frequency-wavenumber representation;

f is the frequency variable;

k is the wavenumber variable.

For spatial information in the resulting f - k representation $V(f, k)$, the spatial component is lost during the transformation. However, it is often desirable to know how the wave number varies along the propagation distance of the wave. In order to preserve and subsequently transform spatial information, a new short spatial two-dimensional Fourier transform was developed to obtain the space-frequency-wave number representation.

This technique can be viewed as a straightforward extension of the short-time Fourier transform to two-dimensional problems, i.e., breaking up the time-space wave field into small segments along the spatial dimension before applying the Fourier transform.

The first step in the numerical implementation of such a technique is that the wave-field data are multiplied by a window function of fixed size. This function is not zero only for a short period in space, but is constant over the entire time dimension.

In a subsequent step, a two-dimensional Fourier transform is applied to the resulting wave-field segments. As the window slides along the spatial dimension, a set of windowed wave-field segments is generated. A two-dimensional Fourier transform is again applied to these segments, resulting in a set of frequency and wavenumber spectra, which are indexed by the window location.

Spatial information can only be preserved by this method. The window space is realized using the two-dimensional Fourier transform technique and has the form

$$S(\bar{x}, f, k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v(t, x) W^*(t, x - \bar{x}) \exp[-i(2\pi f t - kx)]\, dt\, dx \quad , \qquad (21)$$

where

$\bar{x}$ is the retained spatial argument;

$W(t, x)$ is the window function.

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Summary and conclusions.

The global matrix method is robust and stable for any product of frequency and thickness of the laminated sample. However, its use requires a large amount of information on the dispersion of wave packets in each layer. It is shown that the two-dimensional Fourier method allows to detail the spectral characteristics of all modes propagating in a quasi-isotropic laminate of fixed thickness. However, characteristics of wave packet propagation such as the mode content of the wave and how the modes change along the wave path require the use of additional experimental techniques. Analysis of the calculation results indicates the presence of opposite directions of wave packet propagation for each fixed layer. The emergence of new components was recorded, which are created by waves captured in the stratification region when the waves pass above and below the stratification.

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