# KAPITEL 2 / CHAPTER 2 [2]
# ANALYSIS OF THEORETICAL APPROACHES TO CALCULATING THE STRESS STATE OF MULTILAYER INCOMPRESSIBLE COMPOSITES

DOI: 10.30890/2709-2313.2025-45-02-009

## Introduction

Based on the three-dimensional linearized theory of elasticity for prestressed media, a scientific direction has been formed that studies dynamic processes in composites with periodic architecture. In particular, works [1–4] investigate the features of elastic disturbance propagation under conditions of homogeneity of the initial stress-strain state. The phenomenon of wave motion belongs to the fundamental interdisciplinary categories, which allows it to be analysed at various levels of complexity – from theoretical to applied engineering. At the same time, establishing the physical laws of how initial mechanical stresses affect the acoustic properties of bodies is of great practical importance.

The systematization of achievements in this field, as well as a thorough interpretation of previously established facts, are partially covered in critical reviews [1, 2] and in fundamental monographic publications [3–5].

Within the framework of three-dimensional linearized elasticity theory for preloaded media, a scientific approach to the analysis of elastic wave dynamics in periodically structured composites under conditions of homogeneous structure has been developed. The theoretical foundation for these investigations was provided by classical solutions systematized in fundamental works [3–6]. The existing array of scientific results can be classified into three key areas:

1. Homogenization concepts. Works [3, 4, 5] present a methodology for approximating layered composite structures under initial pressure to a model of an equivalent homogeneous medium.

2. Analysis of planar disturbances. Publications [7–9] are devoted to the study of mechanisms for the transmission of plane waves through multilayer systems.

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 ISBN 978-3-98924-129-9

3. Study of curvilinear waves. Studies [10–12] focus on the propagation of torsional and axisymmetric wave processes in the radial directions of periodic composite structures.

The studies described in [9, 11–13] are based on the assumption of ideal adhesion and continuity at phase boundaries. These findings are most comprehensively summarized in monographs [3, 5]. At the same time, the structure of real composites is usually characterized by the presence of various defects on the interlayer surfaces. To analyse how such imperfections correct the dynamics of plane harmonic waves, a detailed study of the limiting interaction scenario – the effect of complete slip of layers – was conducted in [10]. The issue of the transmission of axisymmetric disturbances in layered structures with uniform pre-stress under conditions of sliding contact surfaces was addressed in scientific articles [11, 12].

The results of the review indicate a thorough understanding of the processes of transmission of plane and axisymmetric harmonic disturbances in prestressed layered composites. This issue has been studied in detail for two extreme scenarios of interlayer interaction: unconditional adhesion (full contact) and free mutual displacement (full slippage). It should be noted that for torsional waves, similar calculations are currently limited only to conditions of ideal surface contact.

## 2.1 Formulation of the scientific problem and the chosen research method

The object of analysis is a multilayer composite structure with residual stresses formed by periodically alternating two types of layers. It is assumed that within each layer type, the physical and mechanical parameters of the material and the indicators of the initial stress-strain state are identical.

In the process of scientific analysis, the system of Lagrangian coordinates $y_n \equiv y^n$ is used, which in the conditions of the initial stress state are identical to Cartesian coordinates. The coordinates $r', \theta, y_3$ are also used, which in the base state correspond to the cylindrical system. The Cartesian coordinate system $(y_1, y_2, y_3)$ is

 ISBN 978-3-98924-129-9

oriented so that its third axis is perpendicular to the boundaries separating the structural layers. The components of the layers are modelled as isotropic hyperelastic media with an arbitrary potential deformation energy. If transversely isotropic materials are considered, it is assumed that their axis of symmetry coincides with the direction of the axis $Oy_3$.

The concept of shear waves refers to normal waves whose radial propagation is caused by torsional vibrations of an infinite layered medium. In this case, the initial stress state is assumed to be uniform (homogeneous).

$$u_m^0 = (\lambda_m - 1)x_m; \quad \lambda_m = const. \tag{1}$$

In addition, we assume that within each individual layer, a stress-strain state with signs of axial symmetry arises.

$$S_{11}^{0(j)} = S_{22}^{0(j)} \neq S_{33}^{0(j)}; \quad \sigma_{11}^{0(j)} = \sigma_{22}^{0(j)} \neq \sigma_{33}^{0(j)};$$
$$\varepsilon_{11}^{0(j)} = \varepsilon_{22}^{0(j)}; \quad \lambda_1^{(j)} = \lambda_2^{(j)}; \quad h'^{(j)} = \lambda_3^{(j)} h^{(j)}; \quad j = 1,2. \tag{2}$$

Given the above assumptions, the analysis of displacement parameters that meet the established requirements will be the main focus [3, 4]

$$u_{r'}^{(j)} \equiv 0; \quad u_\theta^{(j)} = u_\theta^{(j)}(r', y_3, \tau); \quad u_3^{(j)} \equiv 0; \quad u_4^{(j)} \equiv p^{(j)} \equiv 0. \tag{3}$$

Under these circumstances, when formulating general solutions for spatial dynamic linearized problems of elasticity theory, in the context of their application to circular cylindrical coordinates, it is advisable to use the following approach

$$\Psi'^{(j)} = \Psi'^{(j)}(r', y_3, \tau); \quad X'^{(j)} \equiv 0. \tag{4}$$

Under the circumstances described, the procedure for calculating displacement indicators $u_\theta^{(j)}$ for each individual layer is based on the analytical dependencies given below [3, 4]

$$u_\theta^{(j)} = -\frac{\partial}{\partial r'} \Psi'^{(j)}. \tag{5}$$

Regarding the components of the stress tensor $Q'^{(j)}$ when $y_3 = const$ $r=r'$, the following calculation formulas were derived

$$Q'^{(j)}_{3\theta} = \kappa'^{(j)}_{3113} \frac{\partial}{\partial y_3} u_\theta^{(j)}. \tag{6}$$

In the system of dependencies (5), the values of the functions are found by solving the equations below.

$$\left(\Delta' + \kappa'^{(j)}_{3113}\kappa'^{(j)-1}_{1221}\frac{\partial^2}{\partial y_3^2} - \rho'^{(j)}\kappa'^{(j)-1}_{1221}\frac{\partial^2}{\partial \tau^2}\right)\Psi'^{(j)} = 0, \quad (7)$$

where $\Delta'_1 = \frac{\partial^2}{\partial r'^2} + \frac{1}{r'}\frac{\partial}{\partial r'}$; $\rho'^{(j)}$ – density of the medium for each structural element (layer) under preliminary loading conditions; $\tau$ denotes the time coordinate. The calculation $\kappa'^{(j)}$ of the components of the tensors κ^'(j) is performed in accordance with the specifics of specific scientific tasks [3, 5, 13].

Therefore, considering the above arguments, the analysis of the propagation of torsion waves in incompressible multilayer composites under initial loads consists in finding solutions to differential equation (7). In this case, it is necessary to adhere to the principles of continuity at the boundaries between structural components and to take into account the conditions of periodicity in accordance with the provisions of Flocke's theory.

Let us examine the features of the radial propagation of torsional waves parallel to the layers of the material. In this situation, based on the methodology [3, 5], to calculate the "real" phase velocity of wave processes in a pre-stressed layered medium, it is advisable to establish the following

$$\Psi'^{(j)}(r', y_3, \tau) = \Psi'^{(j)(0)}(y_3)H_0^{(1)}(r'k)e^{-i\omega\tau}; \quad C = \omega k^{-1}; \quad j = 1,2. \quad (8)$$

In formula (8), $k$ and $\omega$ represent the wave number and cyclic frequency; $C$ determines the "real" phase velocity of torsional oscillations. The mathematical description of waves diverging to infinity is performed using the Hankel function of the first kind of zero order $H_0^{(1)}(x)$, while $\Psi'^{(j)(0)}(y_3)$ denotes the amplitude function. Hereinafter, the symbol (0) marks all parameters related to amplitude values in representations of type (8).

By substituting expression (8) into system (5) to calculate the displacement components, the following analytical dependencies are derived:

$$u_\theta^{(j)}(r', y_3, \tau) = u_\theta^{(j)(0)}(y_3)\frac{\partial}{\partial r'}H_0^{(1)}(r'k)e^{-i\omega\tau};$$

$$u_\theta^{(j)(0)}(y_3) = -\Psi'^{(j)(0)}(y_3). \quad (9)$$

Similarly, by substituting expression (8) into formula (6), to calculate the components of the stress tensor $\tilde{Q}'^{(j)}$ at $y_3 = const$:

$$Q'^{(j)}_{3\theta}(r', y_3, \tau) = Q'^{(j)(0)}_{3\theta}(y_3)\frac{\partial}{\partial r'}H_0^{(1)}(r'k)e^{-i\omega\tau};$$

$$Q'^{(j)(0)}_{3\theta}(y_3) = -\kappa'^{(j)}_{3113}\frac{\partial}{\partial y_3}\Psi'^{(j)(0)}(y_3). \tag{10}$$

By substituting expression (8) into equation (7) to find the amplitude function $\Psi'^{(j)(0)}$, the following dependence was formed

$$\left(\kappa'^{(j)}_{3113}\kappa'^{(j)-1}_{1221}\frac{d^2}{dy_3^2} + \rho'^{(j)}\omega^2\kappa'^{(j)-1}_{1221} - k^2\right)\Psi'^{(j)} = 0. \tag{11}$$

Given that in formulas (8)–(11) all dependencies are based on amplitude parameters, the requirements for continuity at the boundaries of the layers and the conditions of periodicity will also be formulated in terms of amplitudes. To this end, let us consider a pair of adjacent layers. Suppose that the first layer (whose parameters have index 1) is located within the coordinate interval along the $Oy_3$ axis in the region $0 \le y_3 \le h'^{(1)}$, while the second layer (with index 2) occupies the region $-h'^{(2)} \le y_3 \le 0$ on the axis $Oy_3$.

Given that in formulas (8)–(11) all dependencies are based on amplitude parameters, the requirements for continuity at the boundaries of the layers and the conditions of periodicity will also be formulated in terms of amplitudes. To this end, let us consider a pair of adjacent layers. Suppose that the first layer (whose parameters have index 1) is located within the coordinate interval along the axis region, while the second layer (with index 2) occupies the region on the axis.

At $y_3 = 0$, the continuity conditions are satisfied.

$$Q'^{(1)(0)}_{3\theta}(0) = 0; \quad Q^{(2)(0)}_{3\theta}(0) = 0; \tag{12}$$

and frequency

$$Q'^{(1)(0)}_{3\theta}\left(h^{(1)}\right) = 0; \quad Q^{(2)(0)}_{3\theta}\left(-h^{(2)}\right) = 0. \tag{13}$$

Therefore, in the context of analysing an incompressible medium, there is a need to find a solution to the ordinary differential equation (11) that correlates with the provisions of (12) and (13) based on the formalism given in paragraphs (9) and (10).

Based on the methodological approach described in [1], it is advisable to formulate the solution to differential equation (11) in the following form:

$$\Psi'^{(j)(0)}(y_3) = A_5^{(j)} e^{ik\alpha_3^{(j)} y_3} + A_6^{(j)} e^{-ik\alpha_3^{(j)} y_3}; \quad A_n^{(j)} = const. \tag{14}$$

Let us modify expression (14) by introducing specific constants $B_n^{(j)}$ for each structural element, transforming the dependence relative to the central plane of the corresponding layers. With this approach, the mathematical formulation (14) takes the following form:

$$\Psi'^{(1)(0)}(y_3) = B_5^{(1)} e^{ik\alpha_3^{(1)}\left(y_3 - \frac{1}{2}h'^{(1)}\right)} + B_6^{(1)} e^{-ik\alpha_3^{(1)}\left(y_3 - \frac{1}{2}h'^{(1)}\right)};$$

$$\Psi'^{(2)(0)}(y_3) = B_5^{(2)} e^{ik\alpha_3^{(2)}\left(y_3 + \frac{1}{2}h'^{(2)}\right)} + B_6^{(2)} e^{-ik\alpha_3^{(2)}\left(y_3 + \frac{1}{2}h'^{(2)}\right)}. \tag{15}$$

For (14) and (15), due to $\alpha_3^{(j)}$ in (11) denoted

$$\alpha_3^{(j)} = \sqrt{\kappa'^{(j)-1}_{3113}\left(\rho'^{(j)} C^2 - \kappa'^{(j)}_{1221}\right)}; \quad C = \omega k^{-1}. \tag{16}$$

Within the scope of this study, for incompressible layered composites under prestress, it is advisable to decompose the basic problem into two independent cases. The first concerns symmetric torsional waves, where the displacement parameters $u_\theta^{(j)}$ are symmetric about the midplane of each structural element in radial propagation. The second case covers asymmetric torsional waves, for which the indices $u_\theta^{(j)}$ are characterized by asymmetry relative to the center of the layers $Or'$. In further analysis, these types of wave processes will be investigated as separate physical phenomena.

Symmetric torsion waves. In the context of the situation under study, we establish the following analytical relationships for expression (15):

$$B_5^{(j)} = B_6^{(j)}. \tag{17}$$

Formulas (17), (15), and (9) show that $u_\theta^{(j)}$ are symmetric about the center of each layer. Then, from (17), (15), (9), (10), (12), and (13), the conditions of continuity and periodicity coincide.

If imperfect adhesion (loose contact) is observed between the components of the composite, the requirements for field continuity and periodicity conditions take the following mathematical form:

$$B_5^{(1)} k\alpha_3^{(1)} \kappa'^{(1)}_{3113} \sin\frac{1}{2} k\alpha_3^{(1)} h'^{(1)} = 0;$$

$$B_5^{(2)} k\alpha_3^{(2)} \kappa'^{(2)}_{3113} \sin\frac{1}{2} k\alpha_3^{(2)} h'^{(2)} = 0. \qquad (18)$$

Under conditions of loose coupling between layers, the mathematical form of the dispersion equation is presented as follows:

$$\alpha_3^{(1)} \alpha_3^{(2)} \kappa'^{(1)}_{3113} \kappa'^{(2)}_{3113} \sin\frac{1}{2} k\alpha_3^{(1)} h'^{(1)} \sin\frac{1}{2} k\alpha_3^{(2)} h'^{(2)} = 0. \qquad (19)$$

The mathematical structure of equation (19) allows us to find its solutions in analytical form. The four solutions found for expression (19) are written as follows:

$$C_1 = \sqrt{\frac{\kappa'^{(1)}_{1221}}{\rho'^{(1)}}}; \quad C_2 = \sqrt{\frac{\kappa'^{(2)}_{1221}}{\rho'^{(2)}}}; \quad C_3 = \sqrt{\frac{\kappa'^{(1)}_{1221}}{\rho'^{(1)}} + \frac{4\pi^2 n^2 \kappa'^{(1)}_{3113}}{\rho'^{(1)} k^2 h'^{(1)2}}};$$

$$C_4 = \sqrt{\frac{\kappa'^{(2)}_{1221}}{\rho'^{(2)}} + \frac{4\pi^2 n^2 \kappa'^{(2)}_{3113}}{\rho'^{(2)} k^2 h'^{(2)2}}}; \quad n = 0,1.. \qquad (20)$$

Analysis of dependencies (19) and (20) indicates the absence of mechanical interaction between the structural elements of the composite under conditions of mutual sliding. The speed at which symmetric torsional waves propagate within each layer is determined by its physical and mechanical characteristics, geometric thickness, and level of prior stresses. Within the long-wave approximation, the calculation of the propagation velocities of such waves for both layers is based on the first two analytical expressions from system (20).

It should be noted that the initial components in formulas (20) describe the propagation velocity of shear waves in a homogeneous medium in the presence of prior stresses corresponding to the parameters of the first and second layers, respectively.

Let us consider torsional waves of an asymmetric type. For the scenario under study, when constructing a solution in the form (17) for a pair of adjacent structural elements, we establish the following relationships:

$$B_5^{(j)} = -B_6^{(j)}. \qquad (21)$$

Taking into account constraint (21), based on expressions (15) and (9), we can conclude that the indicators $u_\theta^{(j)}$ take on an asymmetric form with respect to the central planes of the corresponding layers. Under these conditions, based on the set of formulas

(21), (15), (9), (10), (12), and (13), the requirements for field continuity are identical to the conditions of periodicity.

If a non-rigid type of contact is implemented between the structural components of the composite, the mathematical formulation of the principles of continuity and periodicity can be summarized as follows:

$$B_5^{(1)} k\alpha_3^{(1)} {\kappa'}_{3113}^{(1)} \cos\frac{1}{2} k\alpha_3^{(1)} h'^{(1)} = 0;$$

$$B_5^{(2)} k\alpha_3^{(2)} {\kappa'}_{3113}^{(2)} \cos\frac{1}{2} k\alpha_3^{(2)} h'^{(2)} = 0. \quad (22)$$

Under conditions of imperfect coupling of structural elements, the dispersion equation can be represented in the following analytical form:

$$\alpha_3^{(1)} \alpha_3^{(2)} {\kappa'}_{3113}^{(1)} {\kappa'}_{3113}^{(2)} \cos\frac{1}{2} k\alpha_3^{(1)} h'^{(1)} \cos\frac{1}{2} k\alpha_3^{(2)} h'^{(2)} = 0. \quad (23)$$

The structure of expression (23) allows us to obtain its solutions in analytical form. The four functions found that satisfy equation (23) are written as follows:

$$C = \sqrt{\frac{{\kappa'}_{1221}^{(1)}}{\rho'^{(1)}}}; \quad C = \sqrt{\frac{{\kappa'}_{1221}^{(2)}}{\rho'^{(2)}}};$$

$$C = \sqrt{\frac{{\kappa'}_{1221}^{(1)}}{\rho'^{(1)}} + \frac{{\kappa'}_{3113}^{(1)}}{\rho'^{(1)}} \frac{\pi^2(1+2n)^2}{k^2 h'^{(1)2}}}; \quad C = \sqrt{\frac{{\kappa'}_{1221}^{(2)}}{\rho'^{(2)}} + \frac{{\kappa'}_{3113}^{(2)}}{\rho'^{(2)}} \frac{\pi^2(1+2n)^2}{k^2 h'^{(2)2}}}; \quad n = 0,1,\ldots \quad (24)$$

Analytical conclusions from expressions (23) and (24) demonstrate that, under conditions of mutual sliding, the mechanical connection between the components of the composite is neutralized. The speed at which asymmetric torsional waves propagate within each individual layer is determined by its physicochemical properties, geometric parameters (thickness), and the level of pre-stress. Within the long-wave approximation, the calculation of phase velocities for each structural element is based on the initial two equations of the system (24). It should be emphasized that these first components in formulas (24) characterize the dynamics of shear waves in a homogeneous medium with corresponding initial loads for the first and second layers separately.

For incompressible layered composites subjected to initial stresses, it is advisable to decompose the basic mathematical model into two autonomous components. The first describes symmetric torsional waves in which the displacement components $u_\theta^{(j)}$ are symmetrically distributed relative to the central planes of the layers during radial motion along the $Or'$ axis. The second component concerns asymmetric torsion waves, where the parameters $u_\theta^{(j)}$ show signs of asymmetry relative to the geometric center of each element. In the following sections, these types of wave processes will be studied separately.

*Let us consider torsional waves of the symmetric type.* Within the limits of the situation under consideration, we establish the following analytical relationships for mathematical expression (24):

$$B_5^{(j)} = B_6^{(j)}. \qquad (25)$$

Based on the analysis of dependencies (25) and (4), it can be stated that the distribution $u_\theta^{(j)}$ is characterized by symmetry with respect to the central plane of each structural element. Under such circumstances, the requirements for field continuity are identical to the conditions of periodicity. If a scenario of non-rigid (imperfect) contact is realized between the phases of a layered composite, the corresponding analytical constraints take the following form:

$$B_5^{(1)} k\alpha_3^{(1)} {\kappa'}_{3113}^{(1)} \sin\frac{1}{2} k\alpha_3^{(1)} h'^{(1)} = 0;$$
$$B_5^{(2)} k\alpha_3^{(2)} {\kappa'}_{3113}^{(2)} \sin\frac{1}{2} k\alpha_3^{(2)} h'^{(2)} = 0. \qquad (26)$$

Under conditions of imperfect mechanical coupling of layers, the mathematical model of the dispersion equation is presented in the following form:

$$\alpha_3^{(1)} \alpha_3^{(2)} {\kappa'}_{3113}^{(1)} {\kappa'}_{3113}^{(2)} \sin\frac{1}{2} k\alpha_3^{(1)} h'^{(1)} \sin\frac{1}{2} k\alpha_3^{(2)} h'^{(2)} = 0. \qquad (27)$$

The mathematical structure of expression (27) allows us to obtain its solutions in analytical form. The four functions found that satisfy equation (27) are presented in the following form:

$$C_1 = \sqrt{\frac{{\kappa'}_{1221}^{(1)}}{\rho'^{(1)}}}; \quad C_2 = \sqrt{\frac{{\kappa'}_{1221}^{(2)}}{\rho'^{(2)}}}; \quad C_3 = \sqrt{\frac{{\kappa'}_{1221}^{(1)}}{\rho'^{(1)}} + \frac{4\pi^2 n^2 {\kappa'}_{3113}^{(1)}}{\rho'^{(1)} k^2 h'^{(1)2}}};$$

$$C_4 = \sqrt{\frac{{\kappa'}_{1221}^{(2)}}{\rho'^{(2)}} + \frac{4\pi^2 n^2 {\kappa'}_{3113}^{(2)}}{\rho'^{(2)} k^2 h'^{(2)2}}}; \quad n = 0,1. \qquad (28)$$

Analytical conclusions from expressions (27) and (28) confirm that, under conditions of mutual sliding, the mechanical connection between the structural components of the composite is neutralized. The speed at which symmetrical torsional waves are transmitted within each layer is determined by its physical and mechanical characteristics, geometric parameters, and the level of previous loads.

Within the long-wave approximation, the calculation of the propagation velocities of such wave processes for both layers is based on the initial two equations of the system (28). It should be noted that these first components in formulas (28) describe the dynamics of shear waves in a homogeneous medium with corresponding initial stresses for the first and second layers separately.

*Let us consider torsional waves of an asymmetric type.* In the context of the scenario under consideration, when constructing a solution in the form (25) for a pair of adjacent structural elements, we establish the following analytical relationships:

$$B_5^{(j)} = -B_6^{(j)}. \quad (29)$$

Given the constraint (29), analysis of expressions (24) and (5) shows that the indicators $u_\theta^{(j)}$ take on an asymmetric form relative to the central plane of the corresponding layers. Under these circumstances, based on the set of formulas, the requirements for field continuity and periodicity conditions are identical. If a scenario of non-rigid interaction is realized between the structural elements of the composite, these conditions acquire the following mathematical representation:

$$B_5^{(1)} k\alpha_3^{(1)} \kappa'^{(1)}_{3113} \cos\frac{1}{2} k\alpha_3^{(1)} h'^{(1)} = 0;$$

$$B_5^{(2)} k\alpha_3^{(2)} \kappa'^{(2)}_{3113} \cos\frac{1}{2} k\alpha_3^{(2)} h'^{(2)} = 0. \quad (30)$$

Under conditions of imperfect mechanical coupling of layers, the mathematical form of the dispersion equation is presented as follows:

$$\alpha_3^{(1)} \alpha_3^{(2)} \kappa'^{(1)}_{3113} \kappa'^{(2)}_{3113} \cos\frac{1}{2} k\alpha_3^{(1)} h'^{(1)} \cos\frac{1}{2} k\alpha_3^{(2)} h'^{(2)} = 0. \quad (31)$$

The mathematical structure of expression (31) allows us to find its solutions in analytical form. The four functions obtained that satisfy equation (31) are presented in the following form:

$$C = \sqrt{\frac{\kappa'^{(1)}_{1221}}{\rho'^{(1)}}}; \quad C = \sqrt{\frac{\kappa'^{(2)}_{1221}}{\rho'^{(2)}}};$$

$$C = \sqrt{\frac{\kappa'^{(1)}_{1221}}{\rho'^{(1)}} + \frac{\kappa'^{(1)}_{3113}}{\rho'^{(1)}} \frac{\pi^2(1+2n)^2}{k^2 h'^{(1)2}}}; \quad C = \sqrt{\frac{\kappa'^{(2)}_{1221}}{\rho'^{(2)}} + \frac{\kappa'^{(2)}_{3113}}{\rho'^{(2)}} \frac{\pi^2(1+2n)^2}{k^2 h'^{(2)2}}}; \quad n = 0,1, \quad (32)$$

The analytical conclusions derived from expressions (31) and (32) confirm the absence of mechanical connection between the structural components of the composite under mutual sliding conditions. The speed at which asymmetric torsional waves propagate within each layer is determined by its physical and mechanical characteristics, geometric parameters, and the level of previous loads.

Within the long-wave approximation, the calculation of the propagation velocities of such wave processes for both components is based on the initial two equations of the system (32). It should be emphasized that the first terms in formulas (24) characterize the dynamics of shear waves in a homogeneous medium with corresponding initial stresses for the first and second layers separately.

## 2.2 Problems of finding an analytical solution for a system consisting of a ring element and two half-spaces under prestress

The analysis and search for solutions to contact problems in elasticity theory remain priority vectors of modern science, especially given the current challenges in engineering. Many applied problems are based on the principles of deformable solid mechanics. These include, in particular: assessment and study of stress-strain indicators of massive foundations and floor slabs under the influence of gravity, as well as the design of cooling towers, water pressure structures, industrial chimneys, and other critical infrastructure facilities and reinforced concrete elements [14-16]. The introduction of innovative materials and the need to optimize their physical and technical properties stimulate increased interest among researchers in in-depth study of processes within the mechanics of deformable media. Numerous professional publications and fundamental monographs [17, 18] provide a thorough analysis of the aspects of contact interaction between plastic, elastic, and viscoelastic objects in the absence of initial mechanical loads. However, modern requirements of engineering and technological industries dictate the need for more complex models. Such approaches must take into account the presence of initial (in particular, technological or residual) stresses, the specifics of surface parameters, the action of friction forces, the level of rigidity of the boundaries, thermal effects, as well as the wear resistance of contact zones [19-21]. Systematization of the results of such research makes it possible to determine the boundary conditions on the contact surfaces of deforming bodies, bringing mathematical models as close as possible to real operating conditions.

According to the developers, the main value of the chosen methodology lies in the possibility of finding a solution to the contact problem in a unified form for both compressible and incompressible media with preliminary loading for any type of elastic potential. This significantly expands the scope of practical application of the derived analytical formulas. Accordingly, in this fragment of the work, the analysis of the interaction of identical half-spaces and an elastic ring under the action of initial stresses (assuming no friction and equality of the roots of the characteristic equation) is carried out on the basis of the theoretical framework [22]. It should be clarified that the parameters describing the ring stamp are marked with the index "3", while the indices '1' and "2" are used for the upper and lower half-spaces, respectively. The hypothesis of the identity of the initial stress-strain states of the stamp and arrays is also accepted. It should be added that in the traditional formulation (without taking into account the initial stress), a similar contact problem was investigated in [23].

*Problem statement.* Consider an elastic ring with height H, which is in a state of pre-stress (Figure. 1). The geometric center of symmetry of the object coincides with the vertical axis $y_3$ of the cylindrical coordinate system ($r$, $\theta$, $y_3$). The deformation process (compression or stretching) of the stamp is caused by the action of two identical half-spaces with initial stresses under the influence of an axisymmetric force characterized by the resultant $P$. The nature of the external influence implies that points on load-free surfaces and areas distant from the zone of connection of the half-spaces with the ring are displaced relative to the base plane $y_3$=0 by a distance ε. Model parameters: $R_1$ and $R_2$ determine the inner and outer radii of the object, respectively, and the half-height of the ring is h=0.5$H$.

External mechanical influence is implemented in such a way that the points of pressure-free surface areas and zones located at a considerable distance from the junction of the half-spaces with the ring element are shifted relative to the base coordinate plane by a certain distance. Geometric parameters of the system: $R_1$ and $R_2$ denote the inner and outer radial dimensions of the die, respectively.

In addition, isotropic elastic objects (both compressible and incompressible) with arbitrary elastic potential are studied within the framework of scientific analysis. These potentials are modelled as functions of algebraic invariants of Green's deformation tensor, which have the property of double continuous differentiability [24]. Additionally, it is assumed that the influence of the stamp causes a slight disturbance of the basic stress state in the half-spaces, which corresponds to certain criteria

$$S_0^{11} = S_0^{22} \neq 0; S_0^{33} = 0; \lambda_1 = \lambda_2 \neq \lambda_3. \quad (33)$$

The study was conducted in the coordinates of the initial deformed state $Oy_i$, with Lagrangian coordinates and relations $y_i = \lambda_i x_i \quad (i = \overline{1,3})$.

In this case, we will limit ourselves to the case of unequal roots ($\xi'^2_2 \neq \xi'^2_3$) of the characteristic (determinant) equation [24].

In the polar cylindrical coordinate system $(r, \theta, z_i)$, where $z_i = v_i^{-1} y_3$, $v_i = \sqrt{n_i}$, $(i = 1,2)$, $n_1 = \xi_2'^2$, $n_2 = \xi_3'^2$. This satisfies the boundary conditions:

1) on the end surfaces of the elastic ring directly in the area of its interaction (contact) $z_i = \pm h / v_i$, where $v_i = \sqrt{n_i}$, $(i = \overline{1,2})$:

$$u_3'^{(i)} - u_3'^{(3)} = \varepsilon,\ Q_{33}'^{(3)} = Q_{33}'^{(i)},\ Q_{3r}'^{(3)} = 0,\ Q_{3r}'^{(i)} = 0\ (R_1 \leq r \leq R_2)\ (i = 1,2), \quad (34)$$

2) on the surfaces of elastic half-spaces outside the zone of their direct connection $z_i = \pm h / v_i$, $(i = \overline{1,2})$:

$$Q_{33}'^{(i)} = 0,\ Q_{3r}'^{(i)} = 0,\ u_3'^{(i)} = 0,\ (0 < r < R_1 \ \text{або}\ r > R_2)\ (i = 1,2), \quad (35)$$

3) on the side edges of the elastic ring-shaped element $r = R_1$ or $r = R_2$:

$$Q_{rr}'^{(3)} = 0,\ Q_{3r}'^{(3)} = 0,\ (|z_i| \leq h / v_i)\ (i = 1,2), \quad (36)$$

The equilibrium equation, which determines the correlation between the vertical displacement of the end surfaces and the total force $P$, is presented as follows:

$$P=-2\pi\int_{R_1}^{R_2} r\,|\,Q'^{(3)}_{33}\,|\,dr,\quad |\,Q'^{(3)}_{33}\,|=|\,Q'^{(3)}_{3r}\,|_{z_i=\pm H/v_i}\quad (i=1{,}2) \qquad (37)$$

Requirement (36) completes the formulation of the linearized spatial problem concerning the contact connection of a cylindrical stamp of finite dimensions and two elastic half-spaces that are in a state of preloading.

*Basic dependencies and solution finding algorithm.* We will estimate the stress-strain state in contact zones for half-spaces with initial stresses based on linearized differential equations [24].

$$Q'^{(i)}_{33}(\rho;0)=\frac{C_{44}(1+m_1)l_1(s-s_0)}{R_1}\int_0^\infty F(\eta)J_0(\eta\rho)d\eta,$$

$$Q'^{(i)}_{3r}(\rho;\xi)\Big|_{\xi=0}=-\frac{C_{44}(1+m_1)}{v_1}\left(\xi-\frac{h}{R_1}\right)\int_0^\infty \eta F(\eta)e^{(\zeta-h/R_1)\eta/v_1}J_1(\eta\rho)d\eta,\Big|_{\xi=0}=0, \qquad (37)$$

$$U'^{(i)}_3(\rho;0)=-\frac{m_1(s_1-s_0)}{v_1}\int_0^\infty \frac{F(\eta)}{\eta}J_0(\eta\rho)d\eta,$$

$$U'^{(i)}_r(\rho;0)=-(1-s_0)\int_0^\infty \frac{F(\eta)}{\eta}J_1(\eta\rho)d\eta, \qquad (38)$$

where $C_{44}=\begin{cases}W'_{1313},\\ \kappa'_{1313}.\end{cases}$

$$\boldsymbol{m_i}=\begin{cases}\dfrac{\omega'_{1111}n_i-\omega'_{3113}}{\omega'_{1133}+\omega'_{1313}};\\ \dfrac{\lambda_1 q_1}{\lambda_3 q_3}\boldsymbol{n_i};\end{cases}\quad \boldsymbol{l_i}=\begin{cases}\dfrac{\omega'_{1331}}{\omega'_{1313}}+\dfrac{\omega'_{1313}-\omega'_{1331}}{\omega'_{1313}}\dfrac{\omega'_{1133}+\omega'_{1313}}{\omega'_{1111}n_i+\omega'_{1133}};\\ \dfrac{\kappa'_{1331}}{\kappa'_{1313}}+\dfrac{\kappa'_{1313}-\kappa'_{1331}}{\kappa'_{1313}}\dfrac{\lambda_3 q_3}{\lambda_3 q_3+\lambda_1 q_1 n_i};\end{cases}\quad \xi=\frac{z_i v_i}{R_1},\quad \zeta_i=$$

$\frac{\xi}{v_i}=\frac{z_i}{R}$, $\eta=\xi R_1$, $(i=1{,}2)$, $s=s_0 l_2 l_1^{-1}$, $s_0=(1+m_2)(1+m_1)^{-1}$,

$s_1=(m_1-1)m_1^{-1}$, $s_2=(v_1 m_2)(v_2 m_1)^{-1}$, $s_3=s_0 v_1 v_2^{-1}$, $F(\eta)$ – desired function, $J_v(x)$ – Bessel functions of real argument.

In a situation where the roots of the characteristic equation are equal ($\xi'^2_2=\xi'^2_3$), the general solution for analyzing the stress-strain state of an elastic ring stamp [24] under the action of initial stresses is established in the following form:

$$\tilde{\chi}=\tilde{\chi}_1+v_i z_i\tilde{\chi}_2,\ (i=1{,}2) \qquad (39)$$

where $\tilde{\chi}_1 = C_0 z_1(3r^2 - 2z_1^2) + A_0(r^2 - 2z_1^2) + \sum_{k=1}^{\infty}\Big[\big(A_k^{(1)} I_0(\gamma_k v_1 r) + A_k^{(2)} K_0(\gamma_k v_1 r)\big)S_1(\gamma_k v_1 z_1) +$

$$+\big(T_k^{(2)} J_0(\alpha_k r) + T_k^{(1)} Y_0(\alpha_k r)\big)S_2(\alpha_k z_1)\Big],$$

$$\tilde{\chi}_2 = C_0 z_1(3r^2 - 2z_1^2) + A_0(r^2 - 2z_1^2) + \sum_{k=1}^{\infty}\Big[\big(B_k^{(1)} I_0(\gamma_k v_1 r) + B_k^{(2)} K_0(\gamma_k v_1 r)\big)S_1(\gamma_k v_1 z_1) +$$

$$+\big(T_k^{(2)} J_0(\alpha_k r) + T_k^{(1)}{}_2 Y_0(\alpha_k r)\big)S_3(\alpha_k z_1)\Big],$$

where $I_v(x)$ – Bessel function of imaginary argument, $S_1 = C_k\, sin(\gamma_k v_1 z_1) + D_k\, cos(\gamma_k v_1 z_1)$, $S_2 = E_k sh(\alpha_k z_1) + F_k ch(\alpha_k z_1)$, $S_3 = N_k sh(\alpha_k z_1) + M_k ch(\alpha_k z_1)$, $C_k$, $D_k$, $E_k$, $F_k$, $N_k$, $M_k$, $A_k^{(1)}, A_k^{(2)}, B_k^{(1)}, B_k^{(2)}, T_k^{(1)}, T_k^{(2)}$ – constant coefficients, $\alpha_k, \gamma_k$ – own values of the task (2) – (5).

Given the above, the stress-strain state in a ring die (for compressible or incompressible media) under the condition of equality of the roots of the characteristic equation and subject to the boundary constraints (2)–(5) will be described in the following analytical form:

$$U_r'^{(3)} = \frac{\varepsilon\omega_2}{R_1}\sum_{k=1}^{\infty}\left\{-2\tilde{A}_0 r + \alpha_k\left[Y_1(\alpha_k r) - \frac{Y_1(\alpha_k R_1)}{J_1(\alpha_k R_1)}J_1(\alpha_k r)\right]\left(\frac{\alpha_k}{v_1}(E_k ch(\alpha_k z_1) + F_k sh(\alpha_k z_1)) + \right.\right.$$

$$\left.\left. + (1+\alpha_k z_1)M_k(ch(\alpha_k z_1) + sh(\alpha_k z_1))\right)\right\}T_k,$$

$$Q_{3r}'^{(3)} = \frac{C_{44}}{v_1}\sum_{k=1}^{\infty}\left\{\alpha_k^2\left[Y_1(\alpha_k r) - \frac{Y_1(\alpha_k R_1)}{J_1(\alpha_k R_1)}J_1(\alpha_k r)\right]\left\langle(1+m_1)\alpha_k\left(\frac{1}{v_1}(E_k sh(\alpha_k z_1) + \right.\right.\right.$$

$$\left.\left.\left. + F_k ch(\alpha_k z_1)) + (sh(\alpha_k z_1) + ch(\alpha_k z_1))M_k\right) + (1+m_2)(sh(\alpha_k z_1) + ch(\alpha_k z_1))M_k\right\rangle\right\}T_k,$$

$$U_3'^{(3)} = \frac{1}{n_1}\sum_{k=1}^{\infty}\left\{\alpha_k\left[Y_1(\alpha_k r) - \frac{Y_1(\alpha_k R_1)}{J_1(\alpha_k R_1)}J_1(\alpha_k r)\right]\left\langle(m_1 h\alpha_k - (m_2-1)v_1)\left(ch(\alpha_k z_1) - \right.\right.\right. \tag{40}$$

$$\left.\left.\left. - sh(\alpha_k z_1)\right)M_k - m_1\alpha_k(F_k ch(\alpha_k z_1) - E_k sh(\alpha_k z_1)) + \frac{4}{n_1}[m_1(\alpha_k^2 - h) + (1-m_2)h]\tilde{A}_0\right\rangle\right\}T_k,$$

$$Q_{33}'^{(3)} = C_{44}\sum_{k=1}^{\infty}\left\{\left\langle\frac{\alpha_k^3}{v_1}(1+m_1)l_1\left[Y_1(\alpha_k r) - \frac{Y_1(\alpha_k R_1)}{J_1(\alpha_k R_1)}J_1(\alpha_k r)\right]\big(F_k ch(\alpha_k z_1) - E_k sh(\alpha_k z_1)\big)T_k + \right.\right.$$

$$\left.\left. + (1+m_2)l_2\alpha_k^2(sh(\alpha_k z_1) - ch(\alpha_k z_1))M_k\right\rangle + 4l_2(1+m_2)\tilde{A}_0\right\}T_k,$$

where $\omega_2 = \frac{v_1^3}{m_1(s_3 - s_2)}$, $\widetilde{A}_0 = \frac{1}{2(1+\widetilde{c}_0 - 2\widetilde{c}_1 + 2\widetilde{c}_2)} \sum_{k=1}^{\infty} \alpha_k^2 \left[ Y_1(\alpha_k R_1) - \frac{Y_1(\alpha_k R_1)}{J_1(\alpha_k R_1)} J_0(\alpha_k R_1) \right]$, $M_k =$

$$\frac{(1+m_1)\left[sh\left(\frac{\alpha_k h}{v_1}\right)+ch\left(\frac{\alpha_k h}{v_1}\right)\right]}{(\tilde{c}_0+\tilde{c}_1)\left[ch\left(\frac{\alpha_k h}{v_1}\right)-sh\left(\frac{\alpha_k h}{v_1}\right)\right]\left\langle(1+m_2)+(1+m_1)\left(\alpha_k-\frac{(\tilde{c}_0-\tilde{c}_1+\alpha_k\tilde{c}_2)}{(\tilde{c}_1+\tilde{c}_0)}\right)\right\rangle},$$

$$F_k = -\frac{v_1}{\alpha_k(\tilde{c}_0+\tilde{c}_1)} - \frac{v_1(\tilde{c}_0-\tilde{c}_1+\alpha_k\tilde{c}_2)}{\alpha_k(\tilde{c}_0+\tilde{c}_1)} M_k,\ E_k = \frac{v_1}{\alpha_k(\tilde{c}_0+\tilde{c}_1)} - \frac{v_1(\tilde{c}_0-\tilde{c}_1+\alpha_k\tilde{c}_2)}{\alpha_k(\tilde{c}_0+\tilde{c}_1)} M_k,$$

$$\tilde{c}_0 = \begin{cases} \omega'_{1111}\omega'^{-1}_{1122}; \\ \lambda_1 q_1 (\lambda_3 q_3)^{-1} (\kappa'_{1133} + \kappa'_{1313}) \kappa'^{-1}_{1122}; \end{cases} \quad \tilde{c}_i = \begin{cases} \lambda_3 \omega'_{1133} m_i \omega'^{-1}_{1122} n_i^{-1}; \\ (\kappa'_{1133} m_i - \kappa'_{3113}) \kappa'^{-1}_{1122} n_i^{-1}; \end{cases}$$

$$(i = \overline{1,2}),$$

where $T_k -$ sought constant.

Based on solution (8) for the stamp model, subject to compliance with the third requirement (2) and the second requirement (3), we determine the eigenvalues of problem (34)–(37). This is done for the scenario of identical roots of the characteristic equation ($n_1$=$n_2$), where the value denotes the solution of the equation:

$$I_1(\gamma_k v_1 R_2) K_1(\gamma_k v_1 R_1) - I_1(\gamma_k v_1 R_1) K_1(\gamma_k v_1 R_2) = 0,$$

$$\alpha_k = \frac{\mu_k}{R_1} \left( J_1(\mu_k) Y_1(\mu_k R_2 R_1^{-1}) - Y_1(\mu_k) J_1(\mu_k R_2 R_1^{-1}) = 0 \right),$$

where $\mu_k$ - solution of the equation $J_1(\mu_k) = 0.$

Based on the boundary conditions (35), we establish that $C_0 = C_k = 0$. Furthermore, after satisfying the first requirement (33), the procedure for determining the unknown functional $F(\eta)$ for expression (38) is implemented through a system of triple integral equations:

$$\int_0^\infty F(\eta) J_0(\eta r) d\eta = 0 \quad (R_2 < r < \infty)$$

$$\int_0^\infty \frac{F(\eta)}{\eta} J_0(\eta r) d\eta = f(r) \quad (R_1 < r < R_2), \tag{41}$$

$$\int_0^\infty F(\eta) J_0(\eta r) d\eta = 0 \quad (0 < r < R_1)$$

where $f(r) = \varepsilon + \frac{\alpha_k}{n_1}\left[\left(\alpha_k h m_1 - v_1(m_1 - 1)\right)\left(ch\left(\frac{\alpha_k h}{v_1}\right) - sh\left(\frac{\alpha_k h}{v_1}\right)\right) M_k - \right.$

$$-m_1\alpha_k\left(F_k ch\left(\frac{\alpha_k h}{v_1}\right)--E_k sh\left(\frac{\alpha_k h}{v_1}\right)\right)\cdot\left[Y_0(\alpha_k r)-\frac{Y_1(\alpha_k R_1)}{J_1(\alpha_k R_1)}J_0(\alpha_k r)\right]T_k+$$

$$+\frac{4}{n_1}[m_1(\alpha_k^2-h)+(1+m_2)h]\tilde{A}_0T_k.$$

Subsequently, based on the system of triple integral equations (41) and the initial boundary conditions (34)–(35), the unknown functional F(η) for expression (38) is found using an infinite set of constant values $T_k$

$$\frac{F(\eta)}{\eta}=\frac{2}{\pi}\left(\varepsilon\psi_0(\eta,0)+\sum_{k=1}^{\infty}\left\langle\frac{\alpha_k}{m_1}T_k\left[\left(\alpha_k hm_1-v_1(m_1-1)\left(ch\left(\frac{\alpha_k h}{v_1}\right)-sh\left(\frac{\alpha_k h}{v_1}\right)\right)\right)M_k\psi_0(\eta,0)-\right.\right.\right.$$

$$\left.-\alpha_k m_1\left(F_k ch\left(\frac{\alpha_k h}{v_1}\right)-E_k sh\left(\frac{\alpha_k h}{v_1}\right)\right)\right]\left(\psi_0(\eta,0)-\frac{Y_1(\alpha_k R_1)}{J_1(\alpha_k R_1)}\psi_0(\eta,\mu_k)\right)\Bigg)+$$

$$\left.+\frac{4}{n_1}[m_1(\alpha_k^2-h)+(1-m_2)h]\widetilde{A}_0T_k\psi_0(\eta,0)\right\rangle, \tag{42}$$

where $\psi(\eta,0)=\frac{\sin\eta}{\eta}$, $\psi(\eta,\mu_k)=\frac{\eta\sin\eta\cos\mu_k-\mu_k\sin\mu_k\cos\eta}{\eta^2-\mu_k^2}$.

Based on the second condition (2), we derive the following relationship:

$$\int_0^{\infty}F(\eta)J_0(\eta\rho)\,d\eta=\frac{R_1}{(1+m_1)l_1(s-s_0)}\sum_{k=1}^{\infty}\left\{\left[Y_0(\alpha_k\rho)-\frac{Y_1(\alpha_k R_1)}{J_1(\alpha_k R_1)}J_0(\alpha_k\rho)\right]\left(\frac{\alpha_k^3(1+m_1)l_1}{v_1}\cdot\right.\right.$$

$$\cdot\left[F_k ch\left(\frac{\alpha_k h}{v_1}\right)--E_k\,sh\left(\frac{\alpha_k h}{v_1}\right)\right]+\alpha_k^2(1+m_2)l_2M_k\left[sh\left(\frac{\alpha_k h}{v_1}\right)-ch\left(\frac{\alpha_k h}{v_1}\right)\right]\Bigg)+4(1+$$

$$\left.+m_2)l_2\tilde{A}_0\right\}T_k \tag{43}$$

where $\rho=\frac{r-R_1}{R_2-R_1}$ – dimensionless coordinate.

Let us perform the integration of expression (11) with respect to the variable $\rho J_0(\mu_n\rho)d\rho$:

$$\int_0^1\rho J_0(\mu_n\rho)\int_0^{\infty}F(\eta)J_0(\eta\rho)d\eta d\rho=\frac{R_1}{(1+m_1)l_1(s-s_0)}\sum_{k=1}^{\infty}\left\{\left[t_{nk}^{(1)}-\frac{Y_1(\alpha_k R_1)}{J_1(\alpha_k R_1)}t_{nk}^{(2)}\right]\left(\frac{\alpha_k^3(1+m_1)l_1}{v_1}\times\right.\right. \tag{44}$$

$$\times\left[F_k ch\left(\frac{\alpha_k h}{v_1}\right)-E_k\,sh\left(\frac{\alpha_k h}{v_1}\right)\right]+\alpha_k^2(1+m_2)l_2M_k\left[sh\left(\frac{\alpha_k h}{v_1}\right)-ch\left(\frac{\alpha_k h}{v_1}\right)\right]\Bigg)+$$

$$\left.+4(1+m_2)l_2\widetilde{A}_0\frac{R_2J_1(\mu_nR_2)-R_1J_1(\mu_nR_1)}{\mu_n}\right\}T_k,$$

 ISBN 978-3-98924-129-9

where $t_{nk}^{(1)}=\frac{R_1\alpha_kJ_0(\mu_nR_1)Y_1(\alpha_kR_1)-R_1\mu_nJ_1(\mu_nR_1)Y_0(\alpha_kR_1)}{\mu_n^2-\alpha_k^2}+$

$$+\frac{R_2\mu_nJ_1(\mu_nR_2)Y_1(\alpha_kR_2)-R_2\alpha_kJ_0(\mu_nR_2)Y_1(\alpha_kR_2)}{\mu_n^2-\alpha_k^2},$$

$$t_{nk}^{(2)}=\frac{R_1\alpha_kJ_0(\mu_nR_1)J_1(\alpha_kR_1)-R_1\mu_nJ_1(\mu_nR_1)J_0(\alpha_kR_1)}{\mu_n^2-\alpha_k^2}+$$

$$+\frac{R_2\mu_nJ_1(\mu_nR_2)J_1(\alpha_kR_2)-R_2\alpha_kJ_0(\mu_nR_2)J_1(\alpha_kR_2)}{\mu_n^2-\alpha_k^2}.$$

When performing calculations using formula (44), the following tabulated values of integrals are used:

$$\int_0^1\rho J_0(\mu_n\rho)\int_0^\infty\eta\psi_0(0,\eta)J_0(\eta\rho)\,d\eta d\rho=\psi_0(0,\mu_k),$$

$$\int_0^\infty\eta\psi(\eta,\mu_k)\int_0^1\rho J_0(\mu_n\rho)J_0(\eta\rho)\,d\rho d\eta=\psi(\mu_n,\mu_k).$$

To determine the unknown coefficients $T_k$, $(k=0,1,2,...)$ that appear in expressions (38), (39), and (42), we derive the corresponding infinite system of equations:

$$\tilde{\alpha}_kT_k+\sum_{n=0}^{\infty}\tilde{\alpha}_{kn}T_n=\tilde{\beta}_k\ \ (k=0,1,2,\dots)\qquad(45)$$

where $\tilde{\beta}_k=-\frac{2\varepsilon\varpi_k}{\pi}$; $\tilde{\alpha}_k=\frac{8}{\pi n_1}(m_1(\alpha_k^2-h)+(1-m_2))\tilde{A}_0\varpi_k$;

$$\tilde{\alpha}_{kn}=\frac{R_1}{(1+m_1)l_1(s-s_0)}\sum_{n=1}^{\infty}\left\{\left[t_{nk}^{(1)}-\frac{Y_1(\alpha_kR_1)}{J_1(\alpha_kR_1)}t_{nk}^{(2)}\right]\left(\frac{\alpha_k^3(1+m_1)l_1}{v_1}\left[F_kch\left(\frac{\alpha_kh}{v_1}\right)-E_ksh\left(\frac{\alpha_kh}{v_1}\right)\right]+\right.\right.$$

$$\left.\left.+\alpha_k^2(1+m_2)l_2M_k\left[sh\left(\frac{\alpha_kh}{v_1}\right)-ch\left(\frac{\alpha_kh}{v_1}\right)\right]\right)+4(1+m_2)l_2\tilde{A}_0\frac{R_2J_1(\mu_nR_2)-R_1J_1(\mu_nR_1)}{\mu_n}\right\}-$$

$$-\frac{2}{\pi}\sum_{n=1}^{\infty}\left\langle\frac{\alpha_k}{m_1}\left[(\alpha_khm_1-v_1(m_1-1))\left(ch\left(\frac{\alpha_kh}{v_1}\right)-sh\left(\frac{\alpha_kh}{v_1}\right)\right)M_k\varpi_k-\right.\right.$$

$$\left.\left.-\alpha_km_1\left(F_kch\left(\frac{\alpha_kh}{v_1}\right)-E_ksh\left(\frac{\alpha_kh}{v_1}\right)\right)\right]\left(\varpi_k-\frac{Y_1(\alpha_kR_1)}{J_1(\alpha_kR_1)}\psi_0(\mu_n,\mu_k)\right)\right\rangle,$$

$$\varpi_k=\frac{(-1)^k\left[R_2^{2(k+1)}\tau_k(R_2)-R_1^{2(k+1)}\tau_k(R_1)\right]}{2^{2n+1}(k+1)k!\,\Gamma(k+1)},$$

where $\tau_k(x)=\sum_{n-0}^{\infty}\frac{x^{2n}\Gamma(2+k)\Gamma(0.5+n)\Gamma(n+1+k)}{n!\Gamma(2+n+k)\Gamma(0.5)\Gamma(1+k)}$.

It should be noted that the parameters of system (13) are determined by the

 ISBN 978-3-98924-129-9

characteristics of the elastic potential, as well as the geometric dimensions (height and inner and outer radii) of the ring stamp. Using the equilibrium equation (37), we determine the correlation between the magnitude of the vertical displacement and the resultant applied force P in the following form:

$$P = 4\pi\varepsilon C_{44}(1 + m_2)l_2(R_1^2 - R_2^2)\tilde{A}_0 T_0.$$

The unknown parameters found as a result of solving the algebraic system (45) allow us to determine the values of displacements and components $T_k$ $(k = 0,1,2,...)$ of the stress tensor both in the arrays of half-spaces and in the elastic ring using analytical expressions (38) and (40).

## 2.3 Contact problems considering the influence of initial stresses on contact characteristics

The scientific significance of contact problems that take into account the influence of initial stresses on interaction parameters is indisputable, since initial mechanical loads are inherent in virtually all structural components and machine assemblies. The particular interest in studying contact processes in bodies with pre-stresses is due to the introduction of innovative synthetic materials capable of withstanding significant initial deformations. In some cases, the deliberate creation of internal stresses (residual or technological) is a sensible strategic decision for leveling the forces that arise during operation, as well as for significantly improving the strength characteristics of structures.

*Formulation of the problem.* Consider a system consisting of two strips made of a homogeneous compressible or incompressible medium with an arbitrary elastic potential. It is assumed that both elements have identical initial (residual) stresses. The strips are connected at finite intervals of the form [-a+2kl; a+2kl], (l>a, k=0±1,±2....), and $k$ takes integer values (0,±1,±2…). The role of connecting elements is performed by periodically located elastic pads, the thickness of which $h$ is insignificant (Figure. 1). The mutual fastening of these elements is implemented at discrete intervals using a system of periodically arranged elastic overlays with a slight thickness index (Figure.

1).

Full contact conditions:

$$u(y_1) = u_1(y_1);\ v(y_1) = u_2(y_1);\ -a < y_1 < a. \qquad (49)$$

The purpose of the analysis is to establish the distribution patterns of contact forces (normal and tangential components) at the interface between elements. Given the periodic nature of the selected modelling method, the effect of the initial stresses under each individual reinforcement element is identical. This allows us to focus on studying only one contact zone - specifically, the one located within the interval [-$a$,$a$].

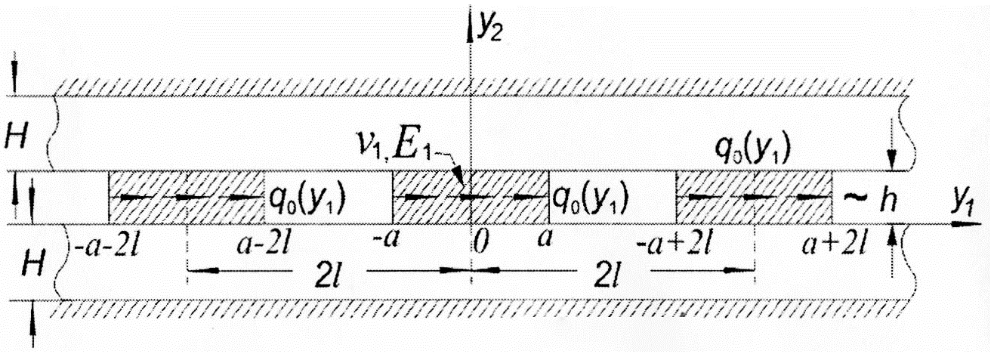


**Figure 1 - Graphical model of the location of connecting elements and applied forces. Here, *H* denotes the thickness of the main strips, and *h* denotes the thickness of the overlays. The elastic properties of the overlay material are characterized by the elastic modulus $E_1$ and Poisson's ratio $\nu_1$. The parameter $g_0(y_1)$ determines the intensity of the periodic load acting in the horizontal direction**

*A source: [25]*

*Solution method.* Taking into account the boundary conditions (2) and analyzing the shift of points on the boundary of the band in the interaction zone $y_1 \in [-a,a]$ at the boundary $y_2$=0, when $y_2$=0, we arrive at the classical system of integro-differential equations of the following form:

$$\frac{d}{dy_1} = \left[ \int_{-a}^{a} h_{11}(|y_1 - t|)p(t)dt + \int_{-a}^{a} h_{12}(y_1 - t)q(t)dt \right] = 0;$$

$$\frac{d}{dy_1} = \left[ \int_{-a}^{a} h_{21}(y_1 - t)p(t)dt + \int_{-a}^{a} h_{22}(|y_1 - t|)q(t)dt \right] =$$


ISBN 978-3-98924-129-9

$$= \frac{1}{E_1 h}\left[2\int_{-a}^{y_1} q(t)dt - Q_0\theta(y_1)\right],$$

at $-a < y_1 > a;$

After performing the substitution $h_{ij}(i,j$=1,2) and the transformation, introducing the function $X(\tau) = \hat{p}(\tau) + i\hat{q}(\tau)$ and replacing (τ=ξ,η) $\delta = \frac{\pi a}{l}$ , we obtain a singular integral equation with a Hilbert kernel:

$$i\beta_1 X(\xi) + \int_{-\delta}^{\delta} X(\eta) ctg\frac{\xi-\eta}{2} d\eta - \int_{-\delta}^{\delta} \tilde{L}_{11}(\xi-\eta) X(\eta) d\eta -$$

$$-i\int_{-\delta}^{\delta} \tilde{L}_{12}(\xi-\eta) X(\eta) d\eta - \int_{-\delta}^{\delta} \tilde{L}_{22}(\xi-\eta) \bar{X}(\eta) d\eta + \beta_2 \int_{-\delta}^{\delta} [X(\eta) - \tilde{X}(\eta)]\, d\eta =$$

$$= i\left[\beta_2 \tilde{Q}_1(\xi) + \lambda_4\right] dt - \frac{\beta_1}{2\pi}; (-\delta < \xi < \delta). \quad (50)$$

For the boundary condition

$$\int_{-\delta}^{\delta} X(\eta)\, d\eta = \frac{i\pi}{l} \quad (51)$$

The solution of equation (3) in the form of a series of Jacobi functions:

$$X(\xi) = w(\xi)\sum_{n=1}^{\infty} \widetilde{X_n}\, P_n^{a,\beta}\left(ctg\frac{\delta}{2} tg\frac{\xi}{2}\right), \quad |\xi| < \delta, \quad (52)$$

where $\left\{P_n^{(\alpha,\beta)}(ctg\frac{\delta}{2} - tg\frac{\xi}{2})\right\}_{n=0}^{\infty}$ – Jacobi polynomials on the interval [-δ;δ] with respect to weight

$$w(\xi) = sec\frac{\xi}{2}\left(sin\frac{\delta-\xi}{2}\right)^{a}\left(sin\frac{\delta+\xi}{2}\right)^{\beta},$$

$$a = -\frac{1}{2} - ia_1; \;\; \beta = -\frac{1}{2} + ia_1; \;\; a_1 = \frac{In(3-4c_{44})}{2\pi}$$

$c_{44}$ – parameter that determines the initial stress state in the strips.

As a result of the transformations, we arrive at a quasi-regular system of linear algebraic equations:

$$l_m\tilde{X}_m + \sum_{n=1}^{\infty}\left[D_{m,n}^{(1)}\tilde{X}_n + D_{m,n}^{(2)}\bar{\tilde{X}}_n\right] = -\left[D_m^{(0)} + D_m^{(1)}X_0 + D_m^{(2)}\bar{X}_0\right], \quad (m = 1,2 \ldots). \quad (53)$$

$l_m, D_{m,n,}^{(1)} D_{m,n}^{(2)} D_m^{(0)}, D_m^{(1)}, D_m^{(2)}, (m = 1,2 \ldots)$ – known quantities that depend on the previously stressed state.

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The computational experiment was performed using two specific types of elastic potentials, namely: harmonic and Bartenev-Khazanovich models (the corresponding results are shown in Figure. 2 and 3).

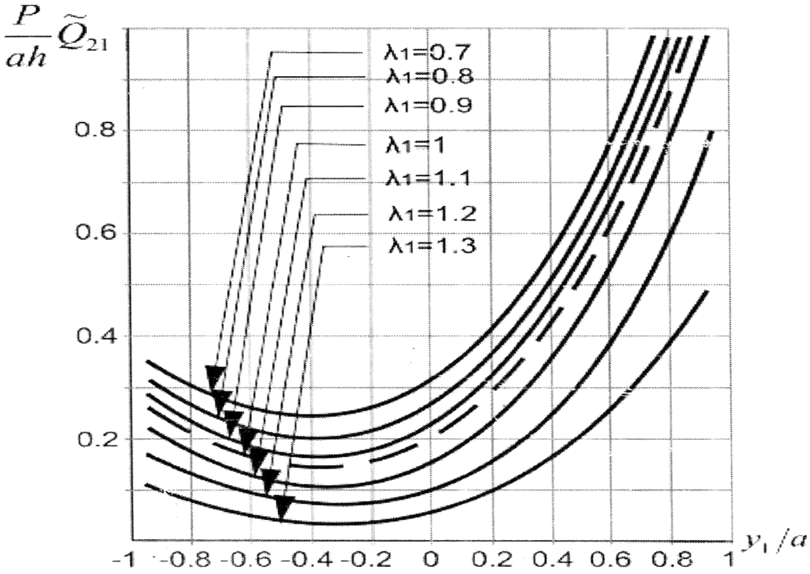


**Figure 2 - Results studying the pattern of distribution of normal and tangential contact stresses along the connection line for harmonic potential (compressed bodies)**

*A source: [25]*

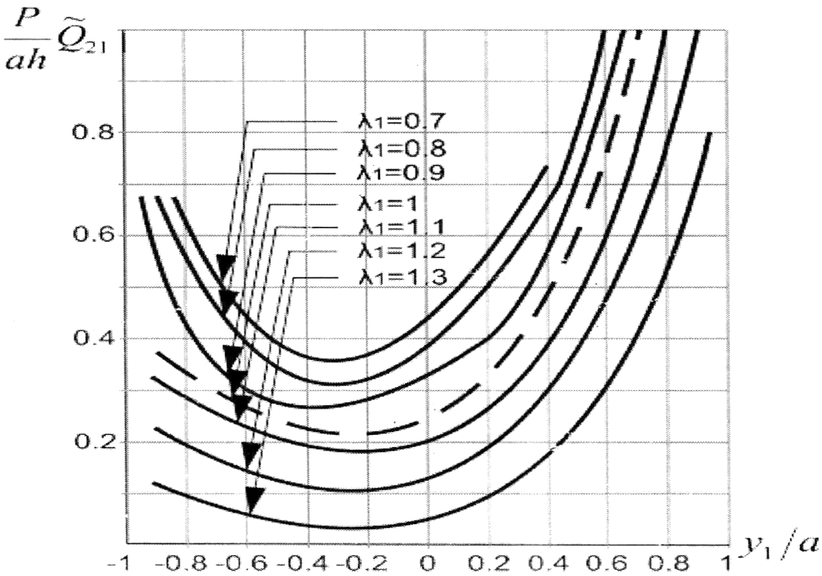


**Figure 3 - Results of the pattern of distribution of normal and tangential contact stresses along the connection line for the potential of Bartenev-Hazanovich (incompressible bodies)**

*A source: [25]*


ISBN 978-3-98924-129-9

## 2.4 Features of the dynamic behaviour of a two-layer half-space under the action of preliminary stresses under conditions of a moving load

Today, several key vectors are actively developing within the framework of the dynamics of elastic media with preliminary stresses. Among them are: analysis of wave transmission mechanisms in bodies with variable geometry [16, 26-28]; the study of the physics of propagating cracks in both homogeneous structures [5, 29, 30] and at the interphase boundaries of heterogeneous materials [31, 32]; and the study of the dynamic response of systems to moving loads [2, 33].

When formulating the basic principles of linearized mechanics of deformable media, the main emphasis is placed on the specifics of deriving the governing equations for materials with elastic and elastic-plastic properties. Verified analytical solutions are obtained using the mathematical tools of complex variable function theory. The method of introducing these variables allows the parameters of initial (residual) stresses to be directly taken into account in the structure of complex arguments.

In this section, based on the theoretical basis [2, 33] and using the integral Fourier transform, a general solution to the problem is derived. The results obtained cover cases of compressible and incompressible media, and also take into account scenarios of rigid and sliding types of contact interaction at the boundary between the layer element and the base.

*Problem statement.* An elastic layer with a thickness $2h$ located on an infinite half-space is being studied. The basic stress-strain state of this system is described by the corresponding components of the displacement vector and the components of the generalized stress tensor as follows:

$$u_j^0 = \delta_{ij}(\lambda_i + 1)x_i; \quad \sigma_{ii}^{*0} \neq 0 \quad (i, j = 1, 2, 3), \tag{54}$$

where $\lambda_i$ – elongations ($\lambda_i = \text{const}$) along the axes of the Lagrangian coordinate system $x_i$, which coincides in the natural state with the Cartesian coordinate system. Let us introduce the Cartesian coordinates $\xi_i$ of the initial deformed state, related to the coordinates $x_i$ by the relations $\xi_i = \lambda_i x_i$.

 ISBN 978-3-98924-129-9

A load moving at a uniform speed v and maintaining invariance with respect to the coordinate $\xi_3$ acts on the unsecured surface of the layer. Such a force causes the occurrence of a plane strain mode in the studied layered array.

In order to find a solution, we apply the apparatus of linearized elasticity theory for compressible media under the influence of preliminary stresses [34]. Based on the assumption that in a coordinate system moving synchronously with an external force, the deformation field is stationary (unchanging over time), we formulate the following propositions: $(y_1, y_2)$, where $y_1 = \xi_1 - \mathrm{v}t$; $y_2 = \xi_2$, the equation of steady motion of a half-space through the function $\chi(y_1, y_2)$ can be written as

$$\left(\eta_1^2 \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right)\left(\eta_2^2 \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right)\chi^{(j)} = 0, \quad j=1,2. \tag{55}$$

The roots $\eta_1$ and $\eta_2$ are determined from the equation

$$\eta^4 + 2A\eta^2 + A_1 = 0, \tag{56}$$

where the coefficients $A$ and $A_1$ for compressible material are determined from the relationships

$$\begin{gathered} 2A\widetilde{\omega}_{2222}\widetilde{\omega}_{2112} = \widetilde{\omega}_{2222}(\widetilde{\omega}_{1111} - \tilde{\rho}v^2) + \widetilde{\omega}_{2112}(\widetilde{\omega}_{1221} - \tilde{\rho}v^2) - \\ -(\widetilde{\omega}_{1122} + \widetilde{\omega}_{1212})^2; \\ 2A_1\widetilde{\omega}_{2222}\widetilde{\omega}_{2112} = (\widetilde{\omega}_{1111} - \tilde{\rho}v^2)(\widetilde{\omega}_{1221} - \tilde{\rho}v^2); \quad \tilde{\rho}\lambda_1\lambda_2\lambda_3 = \rho; \end{gathered} \tag{57}$$

and in the case of incompressible material from the ratios

$$\begin{gathered} 2A\tilde{q}_{22}^2\tilde{\kappa}_{2112} = \tilde{q}_{11}^2\tilde{\kappa}_{2222} + \tilde{q}_{22}^2(\tilde{\kappa}_{1111} - \tilde{\rho}v^2) - 2\tilde{q}_{11}\tilde{q}_{22}(\tilde{\kappa}_{1122} + \tilde{\kappa}_{1212}); \\ 2A_1\tilde{q}_{22}^2\tilde{\kappa}_{2112} = \tilde{q}_{11}^2(\tilde{\kappa}_{1221} - \tilde{\rho}v^2); \quad \tilde{q}_{ij} = \delta_{ij}\lambda_i q_i; \quad \tilde{\rho} = \rho; \end{gathered} \tag{58}$$

In expressions (57) and (58), the parameter $\rho$ denotes the density of the half-space medium in its base (unstressed) state. Let us assume that the dynamic behaviour of the layer can be described by a mathematical model of plate theory, which integrates the effects of rotational inertia and transverse shear deformation. For the case of normal and tangential surface forces acting on the plate, the necessary calculation dependencies are given in [34].

In the coordinate system $(y_1, y_2)$, the equations of plate theory are written as

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$$2h\left(\frac{2G_1}{1-\nu_1}-\rho_1 v^2\right)\frac{\partial^2 u}{\partial y_1^2}-\tau=P_1;$$

$$2h(\kappa G_1-\rho_1 v^2)\frac{\partial^2 w}{\partial y_1^2}-2\kappa G_1 h\frac{\partial\varphi}{\partial y_1}-q=P_2;$$

$$\frac{2h^2}{3}\left(\frac{2G_1}{1-\nu_1}-\rho_1 v^2\delta_0\right)\frac{\partial^2\varphi}{\partial y_1^2}+2\kappa G_1\left(\frac{\partial w}{\partial y_1}-\varphi\right)-\tau=0; \tag{59}$$

In equations (6) $G_1$, $\nu_1$ and $\varrho_1$ are, respectively, the shear modulus, Poisson's ratio, and density of the plate material; $u$ and $w$ are the displacements of the mid-surface of the plate $(y_2=0)$, and $\delta_0$ is a constant that takes the value 1 or 0 depending on whether the rotational inertia of the plate is taken into account or neglected when deriving equations (6); $\varphi$ is the angle of rotation of the plate cross-section; $\varkappa$ is the Timoshenko shear modulus; $q$ and $\tau$ are the normal and tangential stresses acting on the interface between the plate and the half-space, respectively; $P_1$ and $P_2$ are the tangential and normal components of the load on the free surface of the plate.

The bending moment within the plate is calculated using the following analytical relationship:

$$M=\frac{4}{3}\frac{G_1 h^3}{1-\nu_1}\frac{d\varphi}{dy_1} \tag{60}$$

Let us analyse two scenarios of boundary coupling of a plate with a half-space, assuming that $y_2=-h$:

hard contact

$$\tilde{Q}_{21}=\tau;\quad \tilde{Q}_{22}=q;\quad u_2=w;\quad u_1=u+h\varphi; \tag{61}$$

soft contact

$$\tilde{Q}_{21}=0;\quad \tau=0;\quad \tilde{Q}_{22}=q;\quad u_2=w. \tag{62}$$

Consequently, the scientific problem under consideration is transformed into the need to find solutions to equations of motion (55) and (59), supplemented by boundary conditions (61) or (62). Based on the mathematical model of the plate (59) and dependencies (61), (62), the boundary requirements can be expressed in a unified analytical form:


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$$\delta_1\theta_1\left(\frac{d^2u_1}{dy_1^2}-h\frac{d^2\phi}{dy_1^2}\right)-\tilde{Q}_{21}=\delta_1P_1$$

$$\theta_3\frac{d^2u_2}{dy_1^2}-2\kappa hG_1\frac{d\varphi}{dy_1}-\tilde{Q}_{22}=P_2 \qquad (63)$$

$$\theta_2\frac{d^2\varphi}{dy_1^2}+2\kappa G_1\left(\frac{du_2}{dy_1}-\varphi\right)-\delta_1\tilde{Q}_{21}=0$$

Here are the entries:

$$\theta_1=2h\left(\frac{2G_1}{1-\nu_1}-\rho_1v^2\right);\quad \theta_2=\frac{2h^2}{3}\left(\frac{2G_1}{1-\nu_1}-\delta_0\rho_1v^2\right);\quad \theta_3=2h(\kappa G_1-\rho_1v^2).$$

The parameter $\delta_1$ is 1 for rigid contact and 0 for non-rigid contact.

The nature of the functions $\eta_1^2(\mathrm{v})$ and $\eta_2^2(\mathrm{v})$ determines the specifics of the structure of the dynamic equations (55), which directly predetermines the methodology for constructing the desired solutions. The kinematic effect of load displacement on the distribution of roots of expression (56) for models of compressible and incompressible half-space is systematically described in detail in scientific works [35]. Let us present the analytical result of the problem in a unified form, covering scenarios with both multiple and different roots of equation (3).

Stresses, displacements, and displacement velocities in half-space through functions $\chi^{(j)}$ are determined by the formulas [35]:

$$\tilde{Q}_{ij}=\left(\alpha_{ij}^{(12)}\frac{\partial^2}{\partial y_1^2}+\alpha_{ij}^{(22)}\frac{\partial^2}{\partial y_2^2}\right)\frac{\partial\chi^{(2)}}{\partial y_{2-\delta_{ij}}}+\left(\alpha_{ij}^{(11)}\frac{\partial^2}{\partial y_1^2}+\alpha_{ij}^{(21)}\frac{\partial^2}{\partial y_2^2}\right)\frac{\partial\chi^{(1)}}{\partial y_{1+\delta_{ij}}};$$

$$i,j=1{,}2; \qquad (64)$$

$$u_i=-\beta_{i1}^{(i)}\frac{\partial^2\chi^{(i)}}{\partial y_1\partial y_2}+\left(\beta_{i1}^{(j)}\frac{\partial^2}{\partial y_1^2}+\beta_{i2}^{(j)}\frac{\partial^2}{\partial y_2^2}\right)\chi^{(j)};\quad i,j=1{,}2;\quad i\neq j; \qquad (65)$$

$$\dot{u}_i=v\left[\beta_{i1}^{(i)}\frac{\partial^3\chi^{(i)}}{\partial y_1^2\partial y_2}-\left(\beta_{i1}^{(j)}\frac{\partial^2}{\partial y_1^2}+\beta_{i2}^{(j)}\frac{\partial^2}{\partial y_2^2}\right)\frac{\partial\chi^{(j)}}{\partial y_1}\right];\quad i,j=1{,}2;\quad i\neq j; \qquad (66)$$

where in the case of compressible material

$$\alpha_{ii}^{(11)}=\widetilde{\omega}_{ii22}(\widetilde{\omega}_{1111}-\tilde{\rho}v^2)-\widetilde{\omega}_{ii11}(\widetilde{\omega}_{1212}+\widetilde{\omega}_{2211});$$

$$\alpha_{ii}^{(12)}=\widetilde{\omega}_{ii11}(\widetilde{\omega}_{1221}-\tilde{\rho}v^2);\qquad \alpha_{ii}^{(21)}=\widetilde{\omega}_{ii22}\widetilde{\omega}_{2112};$$

$$\alpha_{ii}^{(22)}=\widetilde{\omega}_{ii11}\widetilde{\omega}_{2222}-\widetilde{\omega}_{ii22}(\widetilde{\omega}_{1122}+\widetilde{\omega}_{2121});$$


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$$\alpha_{ij}^{(11)} = \tilde{\omega}_{ij21}(\tilde{\omega}_{1111} - \tilde{\rho}v^2); \quad \alpha_{ij}^{(22)} = \tilde{\omega}_{ij12}\tilde{\omega}_{2222};$$

$$\alpha_{ij}^{(12)} = \tilde{\omega}_{ij12}(\tilde{\omega}_{1221} - \tilde{\rho}v^2) - \tilde{\omega}_{ij21}(\tilde{\omega}_{1122} + \tilde{\omega}_{2121});$$

$$\alpha_{12}^{(21)} = \tilde{\omega}_{ij21}\tilde{\omega}_{2112} - \tilde{\omega}_{ij12}(\tilde{\omega}_{1212} + \tilde{\omega}_{2211});$$

$$\beta_{11}^{(1)} = \beta_{21}^{(2)} = \beta = \tilde{\omega}_{1212} + \tilde{\omega}_{2211}; \quad \beta_{i2}^{(j)} = \tilde{\omega}_{2jj2}; \quad \beta_{i1}^{(j)} = \tilde{\omega}_{1jj1} - \tilde{\rho}v^2;$$

$$i, j = 1,2; \quad i \neq j;$$

and in the case of incompressible material

$$\alpha_{ii}^{(ii)} = (-1)^i \tilde{q}_{jj}^{-1}\tilde{\kappa}_{1212} - \delta_{j2}\tilde{\rho}v^2\tilde{q}_{11}^{-1}; \quad \alpha_{jj}^{(ii)}$$

$$= \tilde{q}_{jj}\tilde{q}_{ii}^{-2}\left(\tilde{\kappa}_{iiii} - \delta_{j2}\tilde{\rho}v^2\right) + \tilde{\kappa}_{jjjj}\tilde{q}_{jj}^{-1} - \tilde{q}_{ii}^{-1}(2\tilde{\kappa}_{1122} + \tilde{\kappa}_{1212});$$

$$\alpha_{ij}^{(12)} = -\tilde{q}_{22}^{-1}\tilde{\kappa}_{ij21}; \quad \alpha_{ij}^{(22)} = \tilde{q}_{11}^{-1}\tilde{\kappa}_{ij12}; \quad \alpha_{ij}^{(11)} = \tilde{q}_{22}^{-1}\tilde{\kappa}_{ij21};$$

$$\alpha_{ij}^{(21)} = -\tilde{q}_{11}^{-1}\tilde{\kappa}_{ij12}; \quad i, j = 1,2; \quad i \neq j;$$

$$\alpha_{22}^{(12)} = \tilde{q}_{22}^{-1}(\tilde{\kappa}_{1221} - \tilde{\rho}v^2); \quad \alpha_{11}^{(12)} = \tilde{q}_{11}\tilde{q}_{22}^{-1}\alpha_{22}^{(12)}; \quad \alpha_{11}^{(21)} = \tilde{q}_{11}^{-1}\tilde{\kappa}_{2112};$$

$$\alpha_{22}^{(21)} = \tilde{q}_{22}\tilde{q}_{11}^{-1}\alpha_{11}^{(21)};$$

$$\beta_{11}^{(1)} = \beta_{12}^{(2)} = \beta = \tilde{q}_{11}^{-1}; \quad \beta_{21}^{(2)} = \beta_{21}^{(1)} = \tilde{q}_{22}^{-1}; \quad \beta_{11}^{(2)} = \beta_{22}^{(1)} = 0;$$

Considering (64) and (65), boundary conditions (63) can be written as

$$\left[\delta_1\theta_1\frac{\partial^2}{\partial y_1^2}\left(\beta_{11}^{(2)}\frac{\partial^2}{\partial y_1^2} + \beta_{12}^{(2)}\frac{\partial^2}{\partial y_2^2}\right) - \frac{\partial}{\partial y_2}\left(\alpha_{21}^{(12)}\frac{\partial^2}{\partial y_1^2} + \alpha_{21}^{(22)}\frac{\partial^2}{\partial y_2^2}\right)\right]\chi^{(2)} -$$

$$-\left[\delta_1\theta_1\beta_{11}^{(1)}\frac{\partial^3}{\partial y_1^2\partial y_2} + \left(\alpha_{21}^{(11)}\frac{\partial^2}{\partial y_1^2} + \alpha_{21}^{(21)}\frac{\partial^2}{\partial y_2^2}\right)\right]\frac{\partial\chi^{(1)}}{\partial y_1} - \delta_1\theta_1 h\frac{\partial^2\phi}{\partial y_1^2} = \delta_1 P_1;$$

$$-2\kappa h G_1\frac{\partial\varphi}{\partial y_1} - \left[\theta_3\beta_{21}^{(2)}\frac{\partial^3}{\partial y_1^2\partial y_2} + \left(\alpha_{22}^{(12)}\frac{\partial^2}{\partial y_1^2} + \alpha_{22}^{(22)}\frac{\partial^2}{\partial y_2^2}\right)\right]\frac{\partial\chi^{(2)}}{\partial y_1} +$$

$$+\left[\theta_3\frac{\partial^2}{\partial y_1^2}\left(\beta_{21}^{(1)}\frac{\partial^2}{\partial y_1^2} + \beta_{22}^{(1)}\frac{\partial^2}{\partial y_2^2}\right) - \frac{\partial}{\partial y_2}\left(\alpha_{22}^{(11)}\frac{\partial^2}{\partial y_1^2} + \alpha_{22}^{(21)}\frac{\partial^2}{\partial y_2^2}\right)\right]\chi^{(1)} = P_2; \qquad (14)$$

$$\theta_2\frac{\partial^2\varphi}{\partial y_1^2} - 2\kappa G_1\varphi - \left[\left(2\kappa G_1\beta_{21}^{(2)} + \delta_1\alpha_{21}^{(12)}\right)\frac{\partial^2}{\partial y_1^2} + \delta_1\alpha_{21}^{(22)}\frac{\partial^2}{\partial y_2^2}\right]\frac{\partial\chi^{(2)}}{\partial y_2} +$$

$$+\left[\left(2\kappa G_1\beta_{21}^{(1)} - \delta_1\alpha_{21}^{(11)}\right)\frac{\partial^2}{\partial y_1^2} + \left(2\kappa G_1\beta_{22}^{(1)} - \delta_1\alpha_{21}^{(21)}\right)\frac{\partial^2}{\partial y_2^2}\right]\frac{\partial\chi^{(1)}}{\partial y_1} = 0.$$

To summarize the above, the scientific problem concerning the steady-state displacement of a two-layer compressible array under the influence of a moving force factor is transformed into a procedure for finding potential functions: $\chi^{(j)}$ and $\varphi$ for boundary conditions (67).

*Constructing a solution in the space of transformations.* The analytical result of the problem is found by applying the apparatus of integral Fourier transforms with respect to the variable $y_1$. Applying the Fourier transform to equations (55), we obtain:

$$\left(\frac{d^2}{dy_2^2}-k^2\eta_1^2\right)\left(\frac{d^2}{dy_2^2}-k^2\eta_2^2\right)\chi^{(j)F}=0;\quad j=1,2.; \tag{68}$$

Let us formulate the general analytical result of the study, taking into account scenarios with both multiple and distinct roots. This will cover various options for contact interaction between the layer and the half-space array at any speed modes of force displacement (in particular, subsonic, transonic, and supersonic). In the transformed Fourier space, the boundary conditions (67) take the following mathematical form:

$$\left(-\alpha_{21}^{(22)}\frac{d^3}{dy_2^3}-k^2\delta_1\theta_1\beta_{12}^{(2)}\frac{d^2}{dy_2^2}+k^2\alpha_{21}^{(12)}\frac{d}{dy_2}+k^4\delta_1\theta_1\beta_{11}^{(2)}\right)\chi^{(2)F}-$$

$$-ik\left(\alpha_{21}^{(21)}\frac{d^2}{dy_2^2}-k^2\delta_1\theta_1\beta_{11}^{(1)}\frac{d}{dy_2}-k^2\alpha_{21}^{(11)}\right)\chi^{(1)F}+k^2\delta_1\theta_1 h\phi^F=\delta_1P_1^F;$$

$$-2ik\kappa hG_1\varphi^F+ik\left(-\alpha_{22}^{(22)}\frac{d^2}{dy_2^2}+k^2\theta_3\beta_{21}^{(2)}\frac{d}{dy_2}+k^2\alpha_{22}^{(12)}\right)\chi^{(2)F}-$$

$$-\left(\alpha_{22}^{(21)}\frac{d^3}{dy_2^3}+k^2\theta_3\beta_{22}^{(1)}\frac{d^2}{dy_2^2}-k^2\alpha_{22}^{(11)}\frac{d}{dy_2}-k^4\theta_3\beta_{21}^{(1)}\right)\chi^{(1)F}=P_2^F; \tag{69}$$

$$(k^2\theta_2+2\kappa G_1)\varphi^F-\left[k^2\left(2\kappa G_1\beta_{21}^{(2)}+\delta_1\alpha_{21}^{(12)}\right)-\delta_1\alpha_{21}^{(22)}\frac{d^2}{dy_2^2}\right]\frac{d\chi^{(2)F}}{dy_2}+$$

$$+ik\left[k^2\left(2\kappa G_1\beta_{21}^{(1)}-\delta_1\alpha_{21}^{(11)}\right)-\left(2\kappa G_1\beta_{22}^{(1)}-\delta_1\alpha_{21}^{(21)}\right)\frac{d^2}{dy_2^2}\right]\chi^{(1)F}=0.$$

The solution to the modified equation (68) will be sought based on the criterion of attenuation of functions at infinity in the following analytical form:

$$\chi^{F(j)}=\left[1-\delta_{j2}(1-\delta_{\eta_1\eta_2})\right]\left\{C_1^{(j)}e^{k_1k\eta_1(y_2+h)}+\left[\delta_{\eta_1\eta_2}(y_2+h)+1-\right.\right.$$

$$\left.\left.-\delta_{\eta_1\eta_2}\right]C_2^{(j)}e^{k_2k\eta_2(y_2+h)}\right\}; \tag{70}$$

where $C_m^{(j)}$ $(j,m=1,2)$ – constant integration,

$$\gamma_j = k_j\eta_j; \quad j = 1,2; \quad \delta_{\eta_1\eta_2} = \begin{cases} 0, & \eta_1 \neq \eta_2 \\ 1, & \eta_1 = \eta_2 \end{cases}; \quad \delta_{j2} = \begin{cases} 0, & j = 1 \\ 1, & j = 2 \end{cases}.$$

Let us introduce integration constants

$$C_1^{(1)} = iC_1; \quad C_2^{(1)} = iC_2; \quad C_1^{(2)} = C_1; \quad C_2^{(2)} = C_2; \qquad (71)$$

Substituting (70) and (71) into (69), we obtain a system of algebraic equations with respect to the unknowns $C_1$, $C_2$ and $\varphi^F$

$$k\left(a_{11}^{(1)} + ka_{11}^{(2)}\right)C_1 + \left(a_{12}^{(1)} + ka_{12}^{(2)} + k^2a_{12}^{(3)}\right)C_2 + a_{13}\phi^F = k^{-2}\delta_1 P_1^F;$$

$$k^2\left(a_{21}^{(1)} + ka_{21}^{(2)}\right)C_1 + k\left(a_{22}^{(1)} + ka_{22}^{(2)} + k^2a_{22}^{(3)}\right)C_2 + a_{23}\varphi^F = -ik^{-1}P_2^F; \qquad (72)$$

$$k^3a_{31}C_1 + k^2\left(a_{32}^{(1)} + ka_{32}^{(2)}\right)C_2 + \left(a_{33}^{(1)} + k^2a_{33}^{(2)}\right)\varphi^F = 0;$$

where $a_{11}^{(1)} = -\gamma_{21}^{(11)} + \delta_{\eta_1\eta_2}\gamma_1\gamma_{21}^{(21)}$; $a_{11}^{(2)} = \delta_1\theta_1\left(\delta_{\eta_1\eta_2}\theta_1^{(21)} - \beta_{11}^{(1)}\gamma_1\right)$;

$$a_{12}^{(1)} = \delta_{\eta_1\eta_2}\left[\gamma_{21}^{(22)} + 2\gamma_2\left(\alpha_{21}^{(21)} - \alpha_{21}^{(22)}\gamma_2\right)\right];$$

$$a_{12}^{(2)} = -\left[\delta_1\delta_{\eta_1\eta_2}\theta_1\left(\beta_{11}^{(1)} + 2\beta_{12}^{(2)}\gamma_2\right) + (1 - \delta_{\eta_1\eta_2})\gamma_{21}^{(12)}\right];$$

$$a_{12}^{(3)} = -\delta_1\theta_1\beta_{11}^{(1)}\gamma_2(1 - \delta_{\eta_1\eta_2}); \quad a_{13} = \delta_1\theta_1 h;$$

$$a_{21}^{(1)} = \gamma_1\gamma_{22}^{(11)} + \delta_{\eta_1\eta_2}\gamma_{22}^{(21)}; \quad a_{21}^{(2)} = \theta_3\left(\theta_2^{(11)} + \delta_{\eta_1\eta_2}\beta_{21}^{(2)}\gamma_1\right);$$

$$a_{22}^{(1)} = \delta_{\eta_1\eta_2}\left[\gamma_{22}^{(12)} - 2\gamma_2\left(\alpha_{22}^{(22)} + \alpha_{22}^{(21)}\gamma_2\right)\right];$$

$$a_{22}^{(2)} = \delta_{\eta_1\eta_2}\theta_3\left(\beta_{21}^{(2)} - 2\beta_{22}^{(1)}\gamma_2\right) + (1 - \delta_{\eta_1\eta_2})\gamma_2\gamma_{22}^{(12)};$$

$$a_{22}^{(3)} = \theta_3\theta_2^{(12)}(1 - \delta_{\eta_1\eta_2}); \quad a_{23} = -2\kappa h G_1;$$

$$a_{31} = 2\kappa G_1\left(\theta_2^{(11)} + \delta_{\eta_1\eta_2}\gamma_1\beta_{21}^{(2)}\right) + \delta_1\left(\delta_{\eta_1\eta_2}\gamma_1\gamma_{21}^{(21)} - \gamma_{21}^{(11)}\right);$$

$$a_{32}^{(2)} = (1 - \delta_{\eta_1\eta_2})\left(2\kappa G_1\theta_2^{(12)} - \delta_1\gamma_{21}^{(12)}\right);$$

$$a_{32}^{(1)} = \delta_{\eta_1\eta_2}\left\{2\kappa G_1\left(\beta_{21}^{(2)} - 2\gamma_2\beta_{22}^{(1)}\right) + \delta_1\left[\gamma_{21}^{(22)} + 2\gamma_2\left(\alpha_{21}^{(21)} - \alpha_{21}^{(22)}\gamma_2\right)\right]\right\};$$

$$a_{33}^{(1)} = -2\kappa G_1; \quad a_{33}^{(2)} = -\theta_2;$$

$$\theta_m^{(kj)} = \beta_{m1}^{(k)} - \beta_{m2}^{(k)}\gamma_j^2; \quad \gamma_{mk}^{(nj)} = \alpha_{mk}^{(1n)} - \alpha_{mk}^{(2n)}\gamma_j^2; \quad j,k,m = 1,2.$$

Solution of system (19)

$$C_j = \frac{\delta_1 P_1^F U_1^{(j)} + iP_2^F U_2^{(j)}}{\Delta(k)}; \quad j = 1,2; \quad \varphi^F = \frac{\delta_1 P_1^F U_1 + iP_2^F U_2}{\Delta(k)}; \qquad (73)$$

 ISBN 978-3-98924-129-9

where $\Delta(k)=k^2(b_0+kb_1+k^2b_2+k^3b_3+k^4b_4+k^5b_5)$;

$$U_j^{(1)}=k^{-1}\left(b_{10}^{(j)}+kb_{11}^{(j)}+k^2b_{12}^{(j)}+k^3b_{13}^{(j)}+k^4b_{14}^{(j)}\right);$$

$$U_j^{(2)}=-\left(b_{20}^{(j)}+kb_{21}^{(j)}+k^2b_{22}^{(j)}+k^3b_{23}^{(j)}\right);\quad U_j=k^2\left(b_{30}^{(j)}+kb_{31}^{(j)}+k^2b_{32}^{(j)}\right);$$

$$j=1,2;$$

$$b_0=a_{33}^{(1)}\left(a_{11}^{(1)}a_{22}^{(1)}-a_{12}^{(1)}a_{21}^{(1)}\right);$$

$$b_1=a_{33}^{(1)}\left(a_{11}^{(2)}a_{22}^{(1)}+a_{22}^{(2)}a_{11}^{(1)}-a_{12}^{(1)}a_{21}^{(2)}-a_{12}^{(2)}a_{21}^{(1)}\right)+a_{23}\left(a_{31}a_{12}^{(1)}-a_{11}^{(1)}a_{32}^{(1)}\right);$$

$$b_2=a_{33}^{(1)}\left(a_{22}^{(2)}a_{11}^{(2)}+a_{22}^{(3)}a_{11}^{(1)}-a_{12}^{(2)}a_{21}^{(2)}-a_{12}^{(3)}a_{21}^{(1)}\right)+$$
$$+a_{23}\left(a_{31}a_{12}^{(2)}-a_{11}^{(1)}a_{32}^{(2)}-a_{11}^{(2)}a_{32}^{(1)}\right)+$$
$$+a_{33}^{(2)}\left(a_{11}^{(1)}a_{22}^{(1)}-a_{12}^{(1)}a_{21}^{(1)}\right)+a_{13}\left(a_{21}^{(1)}a_{32}^{(1)}-a_{31}a_{22}^{(1)}\right);$$

$$b_3=a_{33}^{(2)}\left(a_{11}^{(2)}a_{22}^{(1)}+a_{22}^{(2)}a_{11}^{(1)}-a_{12}^{(1)}a_{21}^{(2)}-a_{12}^{(2)}a_{21}^{(1)}\right)+$$
$$+a_{13}\left(a_{21}^{(1)}a_{32}^{(2)}+a_{21}^{(2)}a_{32}^{(1)}-a_{31}a_{22}^{(2)}\right)+a_{23}\left(a_{31}a_{12}^{(3)}-a_{11}^{(2)}a_{32}^{(2)}\right)+$$
$$+a_{33}^{(1)}\left(a_{22}^{(3)}a_{11}^{(2)}-a_{12}^{(3)}a_{21}^{(2)}\right);$$

$$b_4=a_{33}^{(2)}\left(a_{11}^{(2)}a_{22}^{(2)}+a_{11}^{(1)}a_{22}^{(3)}-a_{12}^{(2)}a_{21}^{(2)}-a_{12}^{(3)}a_{21}^{(1)}\right)+$$
$$+a_{13}\left(a_{21}^{(2)}a_{32}^{(2)}-a_{22}^{(3)}a_{31}\right);\quad b_5=a_{33}^{(2)}\left(a_{22}^{(3)}a_{11}^{(2)}-a_{12}^{(3)}a_{21}^{(2)}\right);$$

$$b_{10}^{(1)}=a_{22}^{(1)}a_{33}^{(1)};\quad b_{11}^{(1)}=a_{22}^{(2)}a_{33}^{(1)}-a_{23}a_{32}^{(1)};$$

$$b_{12}^{(1)}=a_{22}^{(3)}a_{33}^{(1)}+a_{33}^{(2)}a_{22}^{(1)}-a_{23}a_{32}^{(2)};\quad b_{13}^{(1)}=a_{33}^{(2)}a_{22}^{(2)};\quad b_{14}^{(1)}=a_{33}^{(2)}a_{22}^{(3)};$$

$$b_{10}^{(2)}=a_{12}^{(1)}a_{33}^{(1)};\quad b_{11}^{(2)}=a_{12}^{(2)}a_{33}^{(1)};\quad b_{12}^{(2)}=a_{12}^{(3)}a_{33}^{(1)}+a_{33}^{(2)}a_{12}^{(1)}-a_{13}a_{32}^{(1)};$$

$$b_{13}^{(2)}=a_{33}^{(2)}a_{12}^{(2)}-a_{13}a_{32}^{(2)};\quad b_{14}^{(2)}=a_{33}^{(2)}a_{12}^{(3)};$$

$$b_{20}^{(1)}=a_{21}^{(1)}a_{33}^{(1)};\quad b_{21}^{(1)}=a_{21}^{(2)}a_{33}^{(1)}-a_{23}a_{31};\quad b_{22}^{(1)}=a_{21}^{(1)}a_{33}^{(2)};\quad b_{23}^{(1)}=a_{21}^{(2)}a_{33}^{(2)};$$

$$b_{20}^{(2)}=a_{11}^{(1)}a_{33}^{(1)};\quad b_{21}^{(2)}=a_{11}^{(2)}a_{33}^{(1)};\quad b_{22}^{(2)}=a_{11}^{(1)}a_{33}^{(2)}-a_{13}a_{31};\quad b_{23}^{(2)}=a_{11}^{(2)}a_{33}^{(2)};$$

$$b_{30}^{(1)}=a_{21}^{(1)}a_{32}^{(1)}-a_{22}^{(1)}a_{31};\quad b_{31}^{(1)}=a_{21}^{(1)}a_{32}^{(2)}+a_{21}^{(2)}a_{32}^{(1)}-a_{22}^{(2)}a_{31};$$

$$b_{32}^{(1)}=a_{21}^{(2)}a_{32}^{(2)}-a_{22}^{(3)}a_{31};$$

$$b_{30}^{(2)}=a_{11}^{(1)}a_{32}^{(1)}-a_{12}^{(1)}a_{31};\quad b_{31}^{(2)}=a_{11}^{(1)}a_{32}^{(2)}+a_{11}^{(2)}a_{32}^{(1)}-a_{12}^{(2)}a_{31};$$


ISBN 978-3-98924-129-9

$$b_{32}^{(2)} = a_{11}^{(2)} a_{32}^{(2)} - a_{12}^{(3)} a_{31}.$$

Apply the Fourier transform to (60), (64), and (66):

$$\tilde{Q}_{jm}^F = \left(-k^2 \alpha_{jm}^{(12-\delta_{jm})} + \alpha_{jm}^{(22-\delta_{jm})} \frac{d^2}{dy_2^2}\right) \frac{d\chi^{(2-\delta_{jm})F}}{dy_2} +$$

$$+ik\left(-k^2 \alpha_{jm}^{(11+\delta_{jm})} + \alpha_{jm}^{(21+\delta_{jm})} \frac{d^2}{dy_2^2}\right) \chi^{(1+\delta_{jm})F}; \quad j,m = 1,2; \tag{74}$$

$$\dot{u}_j^F = -ikv\left(-k^2 \beta_{j1}^{(m)} + \beta_{j2}^{(m)} \frac{d^2}{dy_2^2}\right) \chi^{(m)F} - k^2 v \beta_{j1}^{(j)} \frac{d\chi^{(j)F}}{dy_2}; \quad j,m = 1,2; \quad i \neq m;$$

$$M^F = \frac{4}{3} \frac{ikG_1 h^3}{1-\nu_1} \varphi^F.$$

Based on provisions (70), (71), and (73), the structure of analytical dependencies (74) can be converted to the following form:

$$\tilde{Q}_{mj}^F = (-i)^{\delta_{mj}} k^2 \Delta^{-1}(k) \left(\delta_1 P_1^F \Gamma_{mj}^{(1)} + i P_2^F \Gamma_{mj}^{(2)}\right);$$

$$\dot{u}_j^F = i^{2-j} v k^2 \Delta^{-1}(k) \left(\delta_1 P_1^F \Gamma_2^{(1)} + i P_2^F \Gamma_2^{(2)}\right); \quad m,j = 1,2; \tag{75}$$

$$M^F = k \Delta^{-1}(k) \left(i \delta_1 P_1^F \Gamma_\varphi^{(1)} - P_2^F \Gamma_\varphi^{(2)}\right);$$

where $\Gamma_{mm}^{(j)} = k\left(\gamma_1 \gamma_{mm}^{(11)} + \delta_{\eta_1\eta_2} \gamma_{mm}^{(21)}\right) U_j^{(1)} e^{k\gamma_1(y_2+h)} - \left\{\delta_{\eta_1\eta_2}\left[2\gamma_2\left(\gamma_2 \alpha_{mm}^{(21)} + \right.\right.\right.$ $\left.\left. + \alpha_{mm}^{(22)}\right) - \gamma_{mm}^{(12)}\right] - k\left\{\delta_{\eta_1\eta_2}(y_2+h)\left(\gamma_2 \gamma_{mm}^{(12)} + \gamma_{mm}^{(22)}\right) + (1 - \right.$ $\left.\left. - \delta_{\eta_1\eta_2}) \gamma_2 \gamma_{mm}^{(12)}\right\}\right\} U_j^{(2)} e^{k\gamma_2(y_2+h)};$

$$\Gamma_{mn}^{(j)} = k\left(\gamma_{mn}^{(11)} - \delta_{\eta_1\eta_2} \gamma_1 \gamma_{mn}^{(21)}\right) U_j^{(1)} e^{k\gamma_1(y_2+h)} +$$

$$+\left\{\delta_{\eta_1\eta_2}\left[2\gamma_2\left(\gamma_2 \alpha_{mn}^{(22)} - \alpha_{21}^{(21)}\right) - \gamma_{mn}^{(22)}\right] + \right.$$

$$\left. +k\left[\delta_{\eta_1\eta_2}(y_2+h)\left(\gamma_{mn}^{(12)} - \gamma_2 \gamma_{mn}^{(22)}\right) + \left(1 - \delta_{\eta_1\eta_2}\right) \gamma_{mn}^{(12)}\right]\right\} U_j^{(2)} e^{k\gamma_2(y_2+h)};$$

$$\Gamma_\varphi^{(j)} = \frac{4}{3} \frac{G_1 h^3 U_j}{(1-\nu_1)}.$$

To summarize, the analytical result of the problem describing the steady-state dynamics of a two-layer elastic array with pre-tension under the influence of a moving force factor in the Fourier transform space is expressed by formula (75). Based on the analysis of expression (75), it can be stated that the parameters describing the stress-strain state of a two-layer elastic array demonstrate unlimited growth when certain critical values are reached $\Delta(k) \to 0$. In a situation where the characteristic equation has real positive multiple roots, there is a possibility of resonance effects occurring [1].

The results of studying the function $\Delta(k)$ for models of compressible and incompressible arrays when varying the conditions of interaction between the plate and the base are described in detail in scientific publications [7, 36, 37]. Based on the data presented in these sources, it can be concluded that the number of critical load displacement modes is directly determined by the level of preliminary tension of the half-space, the elastic properties of the contacting elements, and the specifics of their coupling. The effect of initial stresses on critical speeds is most pronounced in cases where relatively flexible plates are used or when a non-rigid type of contact is implemented. It should be emphasized that the minimum critical speed threshold for a flexible (non-rigid) connection is invariably lower than the corresponding indicator for rigid fixation conditions.

Two models were selected to demonstrate practical application: a compressible array described by an elastic harmonic potential, and an incompressible medium with a Bartenev-Khazanovich potential. In the simulation, the initial state of deformation was assumed to be flat, provided that there was no external pressure on the surface. The presented computational data refer to the case of a concentrated linear force, in which the normal and tangential components are calculated based on the following dependencies:

$$P_1 = P\delta(y_1)cos\alpha; \quad P_2 = P\delta(y_1)\sin\alpha; \quad P = G_1; \tag{77}$$

where $\alpha$ – angle of inclination of the load to the axis $Oy_1$.

## 2.5 Prospects for further research

Systematization of the obtained data indicates that the preliminary tension of the array is a key factor determining the nature of the distribution of dynamic stresses, velocity indicators of displacements in the half-space, and the magnitude of the bending moment in the plate. The nature of this effect varies depending on the distance between the point under study in the multilayer system and the area of direct application of the external force. Numerical indicators of the stress-strain state in the local area of the layered medium are determined by the level of initial stresses, the spatial position of the point, and the specifics of the boundary coupling of the elements.

In the pre-critical high-speed range of load displacement, the rigid fixation mode is characterized by lower stress values, displacement velocities in the array, and bending moment in the plate compared to a flexible (non-rigid) connection. At the same time, in the analyzed parameter range, the intensity of the increase in the amplitudes of the sought values under compression conditions exceeds the similar indicators under tension. The process of dissipation (attenuation) of disturbances as they move away from the zone of force application during preliminary compression is less intense. The effect of initial stresses increases radically in proportion to the increase in the speed of movement of the force factor, which is most pronounced in compression load scenarios. It should be noted that with monolithic contact, the sensitivity of the system to speed regimes and initial stresses is weaker than in the case of a non-rigid connection.

Taking into account rotational inertia within the limits of the considered surface load speeds and values $\lambda_1$ in the case of rigid contact introduces a negligible correction (less than 2.6%), but in the case of non-rigid contact, the difference in results will be very large (up to 30%). It is especially necessary to take into account rotational inertia at $\lambda_1 < 1$ and high load movement speeds.

The analysis demonstrates that intensification of the velocity regime leads to progressive symmetry disruption. At the same time, the frontal wave dissipates much more actively and becomes practically invisible in the supersonic range, although it is not completely eliminated. This effect is probably due to the structural heterogeneity (layering) of the array. With a monolithic (rigid) phase coupling, the leading wave attenuation occurs more forcefully than under conditions of compliant contact.

Four-matrix linear finite elements are used in the process of discretization of the studied areas. Software tools based on the FORTRAN algorithmic language were developed for the numerical implementation of dependencies (67) and (70). This software provides the ability to monitor load dynamics and calculate the corresponding stress increments at arbitrary moments in time.

**Conclusions**

This section presents the analytical results of a study of contact interaction, which is of strategic importance for the modern construction and mechanical engineering industries. In particular, the scientific problem of the pressure of a pair of preloaded half-spaces on an elastic cylindrical ring in a state of initial stress, under the condition of ideal sliding (without friction), has been solved. The mathematical description of the results is presented in the form of series expansions through an infinite set of constants $\chi_k \ (k = 0,1,2,...)$. The latter are calculated by applying the method of reduction to a system of linear algebraic equations.

The results obtained allow us to deepen the theoretical basis in the field of mechanics of deformable media. It is obvious that under conditions of mutual sliding, the mechanical connection between the structural components of the composite is neutralized. The speed at which torsional waves propagate within each layer is determined by its elastic characteristics, geometric dimensions, and the level of previous loads. Within the long-wave approximation, the phase velocities of propagation of both symmetric and asymmetric torsion waves in individual layers coincide with the velocities of shear waves in a homogeneous material with corresponding initial stresses for each layer separately.

Thus, within the scope of this work, an analysis of the mechanisms of torsional wave propagation in incompressible multilayer composites subjected to initial stresses under interlayer sliding conditions has been performed. The processes of radial propagation of torsion waves parallel to the planes of the layers are investigated. The scientific problem is transformed into finding solutions to the differential equation for the amplitude function, taking into account the requirements of continuity at the boundaries between media and the principles of periodicity in accordance with the provisions of Flocke's theory. For cases of symmetry and asymmetry of wave processes, dispersion dependencies are derived and their long-wave approximations are formulated.

Under conditions of interlayer sliding, the mechanical connection between the components of a layered composite is completely eliminated. The speed at which torsional waves are transmitted within each element is determined by the elastic characteristics of the medium, its geometric thickness, and the level of previous loads. Within the long-wave approximation, the phase velocities of both symmetric and asymmetric wave processes in each layer coincide with the velocities of shear waves in a homogeneous array with similar initial stresses for both layers separately.

Analysis of the computational experiment data allows us to formulate a number of key conclusions: the presence of prior stresses significantly modifies the main parameters of contact interaction. This effect is particularly pronounced for incompressible media (in particular, for the Bartenev-Khazanovich potential) compared to models of compressible materials; in the absence of initial loads, the results obtained fully correlate with classical theoretical positions (see Figures. 2 and 3, dotted line); the approximation of initial stresses to critical limit values (for the Bartenev-Khazanovich potential $\lambda_1^{cr}$=3-0.5, where $\lambda_1$ is the elongation coefficient) provokes the emergence of surface instability. In other words, the applicability of the developed model is limited to the range where $\lambda_1^{cr}$>3-0.5.


ISBN 978-3-98924-129-9